Central Limit Theorems for Super-OU Processes

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Abstract

In this paper we study supercritical super-OU processes with general branching mechanisms satisfying a second moment condition. We establish central limit theorems for the super-OU processes. In the small and crtical branching rate cases, our central limit theorems sharpen the corresponding results in the recent preprint of Milos in that the limit normal random variables in our central limit theorems are non-degenerate. Our central limit theorems in the large branching rate case are completely new. The main tool of the paper is the so called "backbone decomposition" of superprocesses.

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1 Introduction

1.1 Model

Throughout this paper, $d \geq 1$ is a positive integer and b is a positive number. We use $\xi = \{\xi_t : t \geq 0\}$ to denote an Ornstein-Uhlenbeck process (OU process, for short) on $\mathbb{R}^d$, that is, a diffusion process with infinitesimal generator

$$L := \frac{1}{2}\sigma^2 \triangle - bx \cdot \triangledown. \tag{1.1}$$

For any $x \in \mathbb{R}^d$, we use Π_x to denote the law of ξ starting from x. The semigroup of ξ will be denoted by $\{T_t : t \geq 0\}$.

Consider a branching mechanism of the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda x} - 1 + \lambda x)n(dx), \quad \lambda > 0, \tag{1.2}$$

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where $\alpha = -\psi'(0+) > 0$, $\beta \geq 0$, and n is a measure on $(0, \infty)$ such that

$$\int_{(0,+\infty)} x^2 n(dx) < +\infty. \tag{1.3}$$

Let $\mathcal{M}_F(\mathbb{R}^d)$ be the space of finite measures on $\mathbb{R}^d$. In this paper we will always assume that $X = \{X_t : t \geq 0\}$ is a super-Ornstein-Uhlenbeck process (super-OU process, for short) with underlying spatial motion ξ and branching mechanism ψ. We will sometimes call X a (ξ, ψ)-superprocess. The existence of such superprocesses is well-known, see, for instance, [12]. X is a Markov branching branching process taking values in $\mathcal{M}_F(\mathbb{R}^d)$. For any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. The total mass of the process X is a continuous-state branching process with branching mechanism ψ. The assumption (1.3) implies that the total mass process of X does not explode. Since we always assume that $\alpha > 0$, X is a supercritical superprocess.

Let $\mathcal{B}_b^+(\mathbb{R}^d)$ be the space of positive, bounded measurable functions on $\mathbb{R}^d$. As usual, $\langle f, \mu \rangle := \int f(x)\mu(dx)$ and $\|\mu\| := \langle 1, \mu \rangle$. Then for every $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$,

$$-\log \mathbb{P}_\mu \left(e^{-\langle f, X_t \rangle} \right) = \langle u_f(\cdot, t), \mu \rangle, \tag{1.4}$$

where $u_f(x,t)$ is the unique positive solution to the equation

$$u_f(x,t) + \Pi_x \int_0^t \psi(u_f(\xi_s, t-s))ds = \Pi_x f(\xi_t). \tag{1.5}$$

In addition, we assume that $\psi(\infty) = \infty$ which implies that the probability of the extinction event $\mathcal{E} := \{\lim_{t\to\infty} \|X_t\| = 0\}$ in strictly in $(0,1)$, see for example the summary at then end of [21, Section 10.2.2]. Since ψ is convex with $\psi(0) = 0, \psi(\infty) = \infty$ and $\psi'(0+) < 0$, ψ has exactly two roots in $[0, \infty)$; let λ^* be the larger one. We have

$$\mathbb{P}_\mu(\lim_{t\to\infty} \|X_t\| = 0) = e^{-\lambda^* \|\mu\|}.$$

Using the expectation formula of $\|X_t\|$ and the Markov property of X, it is not hard to prove that (see Lemma 3.1 for a proof), under $\mathbb{P}_\mu$, the process $W_t = e^{-\alpha t}\|X_t\|$ is a positive martingale. Therefore it converges:

$$W_t \to W_\infty, \quad \mathbb{P}_\mu\text{-a.s.} \quad \text{as } t \to \infty. \tag{1.6}$$

Using the assumption (1.3) we can show that, as $t \to \infty$, W_t also converges in $L^2(\mathbb{P}_\mu)$, so W_∞ is non-degenerate and the second moment is finite. Moreover, we have $\mathbb{P}_\mu(W_\infty) = \|\mu\|$ and $\{W_\infty = 0\} = \mathcal{E}$.

The purpose of this paper is to establish central limit theorems for the super-OU process. More precisely, we want to find A_t and C_t, for suitable test functions f, such that $C_t(\langle f, X_t \rangle - A_t)$ converges to some non-degenerate random variable as $t \to \infty$. It turns out that C_t is determined

by the second moment of $\langle f, X_t\rangle$ which depends on the sign of $\alpha - 2\gamma(f)b$, where $\gamma(f)$ is a quantity to be defined later.

There are many papers studying laws of large numbers for branching processes and superprocesses. For example, see [2, 3, 14] for branching processes, and [16, 13, 23] for superprocesses. For super-OU processes with binary branching mechanism, the following weak law of large numbers was proved in [16]:

$$e^{-\alpha t}\langle f, X_t\rangle \to \langle f, \varphi\rangle W_\infty, \quad \text{in probability} \tag{1.7}$$

where $f \in C_c^+(\mathbb{R}^d)$. When $\langle f, \varphi\rangle = 0$, it is natural to consider central limit theorems for $\langle f, X_t\rangle$, that is, to find a normalization C_t so that $C_t\langle f, X_t\rangle$ converges to a non-degenerate Gaussian random variable as $t \to \infty$. For branching processes, there are already papers dealing with central limit theorems. In 1966, Kesten and Stigum [20] gave a central limit theorem for multidimensional Galton-Watson processes by using the Jordan canonical form of the expectation matrix M. Then in [4, 5, 6], Athreya proved central limit theorems for multi-type continuous time Markov branching processes; the main tools used in [4, 5, 6] are also the Jordan canonical form and the eigenvectors of the matrix M_t, the mean matrix at time t. Recently, central limit theorems for branching OU particle systems and for super-OU processes were established in [1] and [25] respectively. However, the limiting normal random variables in the central limit theorems in [1, 25] (see [1, Theorems 3.2 and 3.6] and [25, Theorems 3 and 4]) may be degenerate (i.e., equal to zero), so the central limit theorems in [1, 25] are not completely satisfactory.

In this paper, we sharpen the results of [25] and establish central limit theorems for super-OU processes which are more satisfactory in the sense that the limiting normal random variables in our results are non-degenerate. The setup of this paper is more general than that of [25] in the sense that we allow a general branching mechanism as opposed to the binary branching mechanism in [25]. The only assumption on the branching mechanism is the second momemnt condition (1.3), which is necessary for central limit theorems.

We mention that we are following Athreya's argument for multitype (finite type) branching processes, also called multidimensional Galton-Watson processes, and show that Athreya's ideas for finite dimensional branching processes also work for super-OU processes, which can be regarded as an infinite dimensional branching process. The main tool of this paper is, similar to that of [25], also the backbone decomposition of supercritical superprocesses, see [8]. The main idea of the backbone decomposition is that a supercritical super-OU process can be constructed from a branching OU process (known as the backbone), in which particles live forever (known as immortal particles). After dressing the backbone with subcritical super-OU processes, we get a measure-valued Markov process, which gives a version of the super-OU process. Since subcritical super-OU process will die out in finite time, we can imagine that the limit behavior of super-OU process is determined by the backbone branching OU process. In this paper we prove that these intuitive

ideas work well. For the precise backbone decomposition, see Section 2.1.

Using a similar argument, we can also sharpen results of [1] and establish central limit theorems for branching OU particle systems which are more satisfactory in the sense that limiting normal random variables are non-degenerate.

1.2 Eigenfunctions of OU processes

Recall that $\{T_t, t \geq 0\}$ is the semigroup of the OU process ξ. It is well known that ξ has an invariant density

$$\varphi(x) = \left(\frac{b}{\pi\sigma^2}\right)^{d/2} \exp\left(-\frac{b}{\sigma^2}\|x\|^2\right). \tag{1.8}$$

Let $L^2(\varphi) := \{h : \int_{\mathbb{R}^d} |h(x)|^2\varphi(x)dx < \infty\}$. For $h_1, h_2 \in L^2(\varphi)$, we define

$$\langle h_1, h_2\rangle_\varphi := \int_{\mathbb{R}^d} h_1(x)h_2(x)\varphi(x)dx.$$

In this subsection, we recall some results on the spectrum in $L^2(\varphi)$ of the operator L defined in (1.1), more details can be found in [24]. For $p = (p_1, p_2, \cdots, p_d) \in \mathbb{Z}_+^d$, let $|p| = \sum_{j=1}^d p_j$ and $p! = \prod_{j=1}^d p_j!$. Recall the Hermite polynomials $\{H_p(x),\ p \in \mathbb{Z}_+^d\}$:

$$H_p(x) = (-1)^{|p|}e^{\|x\|^2}\frac{\partial}{\partial x_1^{p_1}\cdots\partial x_d^{p_d}}(e^{-\|x\|^2}). \tag{1.9}$$

The eigenvalues of L are $\{-mb, m = 0, 1, 2...\}$ and the corresponding eigenspaces A_m are given by

$$A_m := \mathrm{Span}\{\phi_p,\ |p| = m\}, \tag{1.10}$$

where

$$\phi_p(x) = \frac{1}{\sqrt{p!2^{|p|}}}H_p\left(\frac{\sqrt{b}}{\sigma}x\right).$$

In particular, $\phi_{0,0,\cdots,0}(x) = 1$, $\phi_{e_i}(x) = \frac{\sqrt{2b}}{\sigma}x_i$, where e_i stands for the unit vector in the x_i direction. The function ϕ_p is an eigenfunction of L corresponding to the eigenvalue $-|p|b$ and therefore

$$T_t\phi_p(x) = e^{-|p|bt}\phi_p(x). \tag{1.11}$$

Moreover, the eigenfunctions $\{\phi_p(x),\ p \in \mathbb{Z}_+^d\}$ form a complete orthonormal basis for $L^2(\varphi)$. Thus every $f \in L^2(\varphi)$ admits the following $L^2(\varphi)$ expansion:

$$f(x) = \sum_{m=0}^{\infty}\sum_{|p|=m} a_p\phi_p(x), \tag{1.12}$$

where $a_p = \langle f, \phi_p\rangle_\varphi$. Define

$$\gamma(f) := \inf\{n \geq 0 :\ \text{there exists } p \in \mathbb{Z}_+^d \text{ with } |p| = n \text{ such that } a_p \neq 0\}, \tag{1.13}$$

where we use the usual convention $\inf \varnothing = \infty$. Define

$$f_{(s)}(x) = \sum_{\gamma(f)\le m<\alpha/(2b)} \sum_{|p|=m} a_p\phi_p(x), \quad f_{(c)}(x) = \sum_{m=\alpha/(2b)} \sum_{|p|=m} a_p\phi_p(x),$$

and

$$f_{(l)}(x) = f(x) - f_{(s)}(x) - f_{(c)}(x) = \sum_{m>\alpha/(2b)}^{\infty} \sum_{|p|=m} a_p\phi_p(x).$$

In this paper we will use $\mathcal{P}$ to denote the function class

$$\mathcal{P} := \{f \in C(\mathbb{R}^d) : \text{ there exists } \ k \in \mathbb{N} \text{ such that } |f(x)|/\|x\|^k \to 0 \text{ as } \|x\| \to \infty\}.$$

We easily see that $\mathcal{P} \subset L^2(\varphi)$ and for $f \in \mathcal{P}$, there exists $k \in \mathbb{N}$,

$$|f(x)| \lesssim 1 + \|x\|^k,$$

where we used the following notation: for two positive functions f and g, $f(x) \lesssim g(x)$ means that there exists a constant $c > 0$ such that $f(x) \le cg(x)$.

1.3 Main results for super-OU processses

In this subsection we give the main results of this paper. The proofs will be given in the later sections. In the remainder of this paper, whenever we deal with an initial configuration $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, we are implicitly assuming that it has compact support.

1.3.1 Large branching rate: $\alpha > 2b\gamma(f)$

For each $p \in \mathbb{Z}_+^d$, we define

$$H_t^p := e^{-(\alpha-|p|b)t}\langle \phi_p, X_t\rangle, \quad t \ge 0.$$

Then one can show (see Lemma 3.1 below) that, if $\alpha > 2|p|b$, then under $\mathbb{P}_\mu$, H_t^p is a martingale bounded in $L^2(\mathbb{P}_\mu)$, and thus the limit $H_\infty^p := \lim_{t\to\infty} H_t^p$ exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$.

Theorem 1.1 *If $f \in \mathcal{P}$ satisfies $\alpha > 2\gamma(f)b$, then as $t \to \infty$,*

$$e^{-(\alpha-\gamma(f)b)t)}\langle f, X_t\rangle \to \sum_{|p|=\gamma(f)} a_p H_\infty^p, \quad \textit{in } L^2(\mathbb{P}_\mu).$$

Remark 1.2 *When $\gamma(f) = 0$, H_t^0 reduces to W_t, and thus $H_\infty^0 = W_\infty$. Therefore by Theorem 1.1 and the fact that $a_0 = \langle f, \varphi\rangle$, we get that, as $t \to \infty$,*

$$e^{-\alpha t}\langle f, X_t\rangle \to \langle f, \varphi\rangle W_\infty, \quad \textit{in } L^2(\mathbb{P}_\mu).$$

In particular, the convergence also holds in $\mathbb{P}_\mu$-probability, so it implies the results in [16] in the case of super-OU processes. Moreover, by (1.6)*, on $\mathcal{E}^c$, we have*

$$\|X_t\|^{-1}\langle f, X_t\rangle \to \langle f, \varphi\rangle, \quad \textit{in } \mathbb{P}_\mu\textit{-probability.}$$

1.3.2 Small branching rate: $\alpha < 2\gamma(f)b$

Let

$$\sigma_f^2 := A \int_0^\infty e^{\alpha s} \langle (T_s f)^2, \varphi \rangle \, ds, \tag{1.14}$$

where

$$A := \psi^{(2)}(0+) = 2\beta + \int_{(0,\infty)} x^2 n(dx) < \infty. \tag{1.15}$$

In the rest of this paper, A will always stand for this constant.

Theorem 1.3 *If $f \in \mathcal{P}$ satisfies $\alpha < 2\gamma(f)b$, then $\sigma_f^2 < \infty$ and, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, it holds that*

$$\left(e^{-\alpha t} \|X_t\|, \ \frac{\langle f, X_t \rangle}{\sqrt{\|X_t\|}} \right) \xrightarrow{d} (W^*, \ G_1(f)), \quad t \to \infty, \tag{1.16}$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W^* and $G_1(f)$ are independent.

Remark 1.4 *Using the theorem above, we get that if $\alpha < 2\gamma(f)b$, then, under $\mathbb{P}_\mu$, we have*

$$e^{-\alpha t/2} \langle f, X_t \rangle \xrightarrow{d} G_1(f) \sqrt{W_\infty},$$

where W_∞ and $G_1(f)$ are the same as in the theorem above.

1.3.3 The critical case: $\alpha = 2\gamma(f)b$

Define

$$\rho_f^2 := A \sum_{|p| = \gamma(f)} (a_p)^2. \tag{1.17}$$

Theorem 1.5 *If $f \in \mathcal{P}$ satisfies $\alpha = 2\gamma(f)b$, then, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, it holds that*

$$\left(e^{-\alpha t} \|X_t\|, \ \frac{\langle f, X_t \rangle}{t^{1/2} \sqrt{\|X_t\|}} \right) \xrightarrow{d} (W^*, \ G_2(f)), \quad t \to \infty,$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$, $G_2(f) \sim \mathcal{N}(0, \rho_f^2)$. Moreover W^* and $G_2(f)$ are independent.

Remark 1.6 *Using the theorem above, we get that if $\alpha = 2\gamma(f)b$, then, under $\mathbb{P}_\mu$, we have*

$$t^{-1/2} e^{-\alpha t/2} \langle f, X_t \rangle \xrightarrow{d} G_2(f) \sqrt{W_\infty}, \quad t \to \infty,$$

where W_∞ and $G_2(f)$ are the same as in the theorem above.

Remark 1.7 *Note that the limiting normal random variables in our Theorems 1.3 and 1.5 are non-degenerate.*

Remark 1.8 *The results of [25] correspond to the case* $\gamma(f) = 1$ *in the present paper. For the small branching rate case of [25], the* σ_f^2 *in (3.1) there should be (in the notation there)*

$$\sigma_f^2 = 2\beta \int_0^\infty e^{-\alpha s} \langle \varphi, (\mathcal{P}_s^\alpha \widetilde{f}(\cdot))^2 \rangle \, ds,$$

$\widetilde{f}(x) = f(x) - \langle f, \phi \rangle$. *It is easy to check that the sum of the last two parts of [25, (3.1)] is 0, that is*

$$\int_0^\infty \langle \varphi, (-2\beta(\mathcal{P}_s^{-\alpha} \widetilde{f}(\cdot))^2 + 4\alpha\beta u(\cdot, s)) \rangle \, ds = 0,$$

where $u(x,s) = \int_0^s (\mathcal{P}_{s-u}^{-\alpha}[(\mathcal{P}_u^{-\alpha} \widetilde{f}(\cdot))^2](x) \, du$. *Furthermore, there is an extra factor* β/α *on the right side of [25, (3.1)] which should not be there. In the critical branching case of [25], there is also an extra factor* β/α *on the right side of [25, (3.2)] which should not be there. The correct form of (3.2) there should be (in the notation of [25])*

$$\sigma_f^2 = 2\beta \int_{\mathcal{R}^d} (x \circ \langle grad(f), \varphi \rangle)^2 \varphi(x) dx.$$

With these minor corrections, the results of [25] coincide with our Theorems 1.1, 1.3 and 1.5 when $\gamma(f) = 1$.

Combining Theorems 1.1, 1.3 and 1.5, we have the following expansion of $\langle f, X_t \rangle$: for any $f \in \mathcal{P}$,

$$\begin{aligned} \langle f, X_t \rangle &= \sum_{\gamma(f) \le m < \frac{\alpha}{2b}} \sum_{|p|=m} a_p e^{-(\alpha - mb)t} \langle \phi_p, X_t \rangle \cdot e^{(\alpha - m)bt} \\ &\quad + \sum_{|p| = \frac{\alpha}{2b}} a_p t^{-1/2} e^{-(\alpha/2)t} \langle \phi_p, X_t \rangle \cdot \sqrt{t} e^{(\alpha/2)t} + \langle f_{(l)}, X_t \rangle \\ &= \sum_{\gamma(f) \le m < \frac{\alpha}{2b}} \sum_{|p|=m} a_p U_p(t) \cdot e^{(\alpha - m)bt} + \sum_{|p| = \frac{\alpha}{2b}} a_p U_p(t) \cdot \sqrt{t} e^{\alpha t/2} + \langle f_{(l)}, X_t \rangle, \end{aligned} \tag{1.18}$$

where

$$U_p(t) = \begin{cases} e^{-(\alpha - |p|b)t} \langle \phi_p, X_t \rangle, & |p| < \frac{\alpha}{2b}, \\ t^{-1/2} e^{-\alpha t/2} \langle \phi_p, X_t \rangle, & |p| = \frac{\alpha}{2b}. \end{cases}$$

Further, if $|p| < \frac{\alpha}{2b}$, then $U_p(t) = H_t^p$ converges to H_∞^p, $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$; if $|p| = \frac{\alpha}{2b}$, $U_p(t)$ converges in law to $G_2(\phi_p)\sqrt{W_\infty}$ with $G_2(\phi_p) \sim \mathcal{N}(0, A)$; $e^{-(\alpha/2)t} \langle f_{(l)}, X_t \rangle$ converges in law to $G_1(f_{(l)})\sqrt{W_\infty}$.

1.3.4 Further results in the large branching rate case

In this subsection we give two central limit theorems for the case $\alpha > 2\gamma(f)b$. Define

$$H_\infty := \sum_{\gamma(f)\le m<\alpha/(2b)} \sum_{|p|=m} a_p H_\infty^p. \tag{1.19}$$

Let

$$\beta_{f_{(s)}}^2 := A \sum_{\gamma(f)\le m<\alpha/(2b)} \frac{1}{\alpha - 2mb} \sum_{|p|=m} a_p^2, \tag{1.20}$$

In Section 3.3 we will see that $\beta_{f_{(s)}}^2 = \langle \mathbf{V}ar_{\delta_x} H_\infty,\ \varphi\rangle$.

Theorem 1.9 *If $f \in \mathcal{P}$ satisfies $\alpha > 2\gamma(f)b$ and $f_{(c)} = 0$, then $\sigma_{f_{(l)}}^2 < \infty$. Under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, it holds that, as $t \to \infty$,*

$$\left(e^{-\alpha t}\|X_t\|,\ \|X_t\|^{-1/2}\left(\langle f, X_t\rangle - \sum_{\gamma(f)=m<\alpha/(2b)} e^{(\alpha-mb)t} \sum_{|p|=m} a_p H_\infty^p\right)\right) \xrightarrow{d} (W^*,\ G_3(f)), \tag{1.21}$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$, and $G_3(f) \sim \mathcal{N}(0, \sigma_{f_{(l)}}^2 + \beta_{f_{(s)}}^2)$. Moreover, W^* and $G_3(f)$ are independent.

Remark 1.10 *If $\alpha > 2|p|b$, then under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, it holds that, as $t \to \infty$,*

$$\left(e^{-\alpha t}\|X_t\|,\ \frac{\left(\langle \phi_p, X_t\rangle - e^{(\alpha-|p|b)t} H_\infty^p\right)}{\|X_t\|^{1/2}}\right) \xrightarrow{d} (W^*,\ G_3), \tag{1.22}$$

where $G_3 \sim \mathcal{N}(0, \frac{A}{\alpha - 2|p|b})$. In particular, for $|p| = 0$, we have

$$\left(e^{-\alpha t}\|X_t\|,\ \frac{\|X_t\| - e^{\alpha t}W_\infty}{\sqrt{\|X_t\|}}\right) \xrightarrow{d} (W^*,\ G_3), \quad t \to \infty,$$

where $G_3 \sim \mathcal{N}(0, \frac{A}{\alpha})$

Remark 1.11 *Using the theorem above, we get that if $\alpha > 2\gamma(f)b$ and $f_{(c)} = 0$, then under $\mathbb{P}_\mu$, we have, as $t \to \infty$,*

$$\left(e^{-\alpha t}\|X_t\|,\ e^{-(\alpha/2)t}\left(\langle f, X_t\rangle - \sum_{\gamma(f)\le m<\alpha/(2b)} e^{(\alpha-mb)t} \sum_{|p|=m} a_p H_\infty^p\right)\right) \xrightarrow{d} (W_\infty,\ \sqrt{W_\infty}G_3(f)),$$

where $G_3(f)$ is the same as in the theorem above.

Theorem 1.12 *If $f \in \mathcal{P}$ satisfies $f_{(c)} \neq 0$, then, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, it holds that, as $t \to \infty$,*

$$\left(e^{-\alpha t}\|X_t\|,\ t^{-1/2}\|X_t\|^{-1/2}\left(\langle f, X_t\rangle - \sum_{\gamma(f)\le m<\alpha/(2b)} e^{(\alpha-mb)t}\sum_{|p|=m} a_p H_\infty^p\right)\right) \stackrel{d}{\to} (W^*,\ G_4(f)), \tag{1.23}$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$, and $G_4(f) \sim \mathcal{N}(0, A\sum_{|p|=\alpha/2b}(a_p)^2)$. Moreover, W^* and $G_4(f)$ are independent.

Remark 1.13 *Note that the limiting normal random variables in our Theorems 1.9 and 1.12 are non-degenerate.*

2 Preliminary

2.1 Backbone decomposition of super-OU processes

In this subsection, we recall the backbone decomposition of [8]. Define another branching mechanism ψ^* by

$$\begin{aligned}\psi^*(\lambda) &= \psi(\lambda+\lambda^*)\\ &= \alpha^*\lambda + \beta\lambda^2 + \int_{(0,\infty)}(e^{-\lambda x} - 1 + \lambda x)e^{-\lambda^* x}n(dx),\end{aligned} \tag{2.1}$$

where

$$\alpha^* = -\alpha + 2\beta\lambda^* + \int_{(0,\infty)}(1-e^{-\lambda^* x})xn(dx).$$

It is easy to see that $\alpha^* = (\psi^*)'(0+) = \psi'(\lambda^*) > 0$. So the (ξ,ψ^*)-superprocess is subcritical. Note that it follow from (2.1) that the measure n^* associated with ψ^* is $e^{-\lambda^* x}n(dx)$, thus for any $n \in \mathbb{N}$, $\int_0^\infty x^n n^*(dx) < \infty$. It follows from [8, Lemma 2] that the (ξ,ψ)-superprocess conditioned on $\mathcal{E}$ has the same law as the (ξ,ψ^*)-superprocess. Let $\mathbb{P}^*_\mu$ be the law of the (ξ,ψ^*)-superprocess with initial configuration μ, and define

$$u^*_f(x,t) = -\log \mathbb{P}^*_{\delta_x}(e^{-\langle f, X_t\rangle}).$$

It was shown in [11] that one can associate with $\{\mathbb{P}^*_{\delta_x} : x \in \mathbb{R}^d\}$ a family of measures $\{\mathbb{N}^*_x : x \in \mathbb{R}^d\}$, defined on the same measurable space as the probabilities $\{\mathbb{P}^*_{\delta_x} : x \in \mathbb{R}^d\}$ and satisfying

$$\mathbb{N}^*_x(1-e^{-\langle f, X_t\rangle}) = -\log \mathbb{P}^*_{\delta_x}(e^{-\langle f, X_t\rangle}) = u^*_f(x,t), \tag{2.2}$$

for all $f \in \mathcal{B}^+_b(\mathbb{R}^d)$ and $t \ge 0$. Intuitively speaking, the branching property implies that $\mathbb{P}^*_{\delta_x}$ is an infinitely divisible measure on the path space of X, that is to say, the space of measure-valued cadlag functions, $\mathbb{D}([0,\infty)\times\mathcal{M}_F(\mathbb{R}^d))$, and (2.2) is a 'Lévy-Khinchine' formula in which $\mathbb{N}^*_x$ plays the role

of its ‘Lévy measure’. The measures $\{\mathbb{N}_x^* : x \in \mathbb{R}^d\}$ will play a crucial role in the forthcoming analysis.

Let $\mathcal{M}_a(\mathbb{R}^d)$ be the space of finite atomic measures on $\mathbb{R}^d$. For $\nu \in \mathcal{M}_a(\mathbb{R}^d)$, let $Z = (Z_t : t \geq 0)$ be a branching OU-process with initial configuration ν. $\{Z_t, t \geq 0\}$ is an $\mathcal{M}_a(\mathbb{R}^d)$-valued Markov process in which individuals, from the moment of birth, live for an independent and exponential distributed period of time with parameter α^* during which they move according to the OU-process issued from their position of birth and at death they give birth at the same position to an independent number of offspring with distribution $(p_n : n \geq 0)$, where $p_0 = p_1 = 0$ and for $n \geq 2$,

$$p_n = \frac{1}{\lambda^* \alpha^*} \left\{ \beta (\lambda^*)^2 \mathbf{1}_{\{n=2\}} + (\lambda^*)^n \int_{(0,\infty)} \frac{x^n}{n!} e^{-\lambda^* x} n(dx) \right\}.$$

The generator of Z is given by

$$F(s) \quad = \quad \alpha^* \sum_{n \geq 0} p_n (s^n - s) = \frac{1}{\lambda^*} \psi(\lambda^*(1-s)). \tag{2.3}$$

Z is refereed as the (ξ, F)-backbone in [8]. Moreover, when referring to individuals in Z we will use the classical Ulam-Harris notation so that every particle in Z has a unique label, see [18]. Let $\mathcal{T}$ be the set of labels of individuals realized in Z. Let $|Z_t|$ be the number of particles alive at time t. For each individual $u \in \mathcal{T}$ we shall write τ_u and σ_u for its birth and death times respectively and $\{z_u(r) : r \in [\tau_u, \sigma_u]\}$ for its spatial trajectory. Now we describe three kinds of immigrations along the backbone Z as follows.

1. **Continuous immigration:** The process $I^{\mathbb{N}^*}$ is measure-valued on $\mathbb{R}^d$ such that

   $$I_t^{\mathbb{N}^*} := \sum_{u \in \mathcal{T}} \sum_{u \wedge \tau_u < r \leq t \wedge \sigma_u} X_{t-r}^{(1,u,r)} ,$$

   where, given Z, independently for each $u \in \mathcal{T}$ with $\tau_u < t$, the processes $X_\cdot^{(1,u,r)}$ are independent copies of the canonical process X, immigrated along the space-time trajectory $\{(z_u(r), r) : r \in (\tau_u, t \wedge \sigma_u]\}$ with rate $2\beta \mathrm{d}r \times \mathrm{d}\mathbb{N}_{z_u(r)}^*$.

2. **Discontinuous immigration:** The processes $I^{\mathbb{P}^*}$ is measure-valued on $\mathbb{R}^d$ such that

   $$I_t^{\mathbb{P}^*} := \sum_{u \in \mathcal{T}} \sum_{t \wedge \tau_u < r \leq t \wedge \sigma_u} X_{t-r}^{(2,u,r)} ,$$

   where, given Z, independently for each $u \in \mathcal{T}$ with $\tau_u < t$, the processes $X_\cdot^{(2,u,r)}$ are independent copies of the canonical process X, immigrated along the space-time trajectory $\{z_u(r) : r \in (\tau_u, t \wedge \sigma_u]\}$ with rate $\mathrm{d}r \times \int_{y \in (0,\infty)} y e^{-\lambda^* y} n(dy) \times \mathrm{d}\mathbb{P}_{y\delta_{z_u(r)}}^*$.

3. **Branching point biased immigration:** The process I^η is measure-valued on $\mathbb{R}^d$ such that

$$I_t^\eta = \sum_{u\in\mathcal{T}} \mathbf{1}_{\sigma_u\le t} X^{(3,u)}_{t-\sigma_u}\ ,$$

where, given Z, independently for each $u\in\mathcal{T}$ with $\sigma_u\le t$, the processes $X^{(3,u)}_\cdot$ are independent copies of the canonical process X issued at time σ_u with law $\mathbb{P}^*_{Y_u\delta_{z_u(\sigma_u)}}$ where, given u has $n\ge 2$ offspring, the independent random variable Y_u has distribution $\eta_n(z_u(r),\mathrm{d}y)$, where

$$\eta_n(dy)=\frac{1}{p_n\lambda^*\alpha^*}\left\{\beta(\lambda^*)^2\delta_0(dy)\mathbf{1}_{\{n=2\}}+(\lambda^*)^n\frac{y^n}{n!}e^{-\lambda^* y}n(dy)\right\}.$$

Now we define another $\mathcal{M}_F(\mathbb{R}^d)$-valued process $I=\{I_t:t\ge 0\}$ by

$$I := I^{\mathbb{N}^*} + I^{\mathbb{P}^*} + I^\eta\ ,$$

where the processes $I^{\mathbb{N}^*}=\{I_t^{\mathbb{N}^*}:t\ge 0\}$, $I^{\mathbb{P}^*}=\{I_t^{\mathbb{P}^*}:t\ge 0\}$ and $I^\eta=\{I_t^\eta:t\ge 0\}$, conditioned on Z, are independent of each other. We denote the law of I by $\mathbb{Q}_\nu$. Recall that ν is the initial configuration of Z.

For $\mu\in\mathcal{M}_F(\mathbb{R}^d)$, let $\widetilde{X}$ be an independent copy of X under $\mathbb{P}^*_\mu$ and be independent of I. Then we define a measure-valued process $\Lambda=\{\Lambda_t:t\ge 0\}$ by

$$\Lambda=\widetilde{X}+I. \tag{2.4}$$

Note that Z, $\widetilde{X}$ and the three immigration processes above are defined on the same probability space. We denote the law of Λ by $\mathbf{P}_{\mu\times\nu}$. When ν is a Poisson random measure with intensity measure $\lambda^*\mu$, then we write this law by $\mathbf{P}_\mu$. The following result is proved in [8].

Proposition 2.1 *For any $\mu\in\mathcal{M}_F(\mathbb{R}^d)$, the process $(\Lambda,\mathbf{P}_\mu)$ is Markovian and has the same law as $(X,\mathbb{P}_\mu)$.*

We will need the following σ-fields later on:

$$\mathcal{F}_t = \sigma(\Lambda_s, s\le t),\quad t\ge 0, \tag{2.5}$$

$$\mathcal{G}_t = \sigma(\Lambda_s, Z_s, s\le t),\quad t\ge 0. \tag{2.6}$$

2.2 Moments

Now we use Laplace transforms to calculate the moments of X. We will omit some details, for these omitted details, see [12]. For any $f\in\mathcal{P}$, we define

$$u_f(x,t,\theta)=-\log\mathbb{P}_{\delta_x}(e^{-\langle\theta f, X_t\rangle}),$$

then

$$u_f(x,t,\theta)+\Pi_x\int_0^t \psi(u_f(\xi_s,t-s,\theta))ds=\theta\Pi_x f(\xi_t). \tag{2.7}$$

Differentiating both sides of (2.7) with respect to θ, we get

$$\begin{aligned}
u_f^{(1)}(x,t,0) &= e^{-\psi'(0+)t}T_t f(x), && (2.8)\\
u_f^{(2)}(x,t,0) &= -\psi^{(2)}(0+)\int_0^t e^{-\psi'(0+)(t-s)}T_{t-s}[u_f^{(1)}(\cdot,s,0)]^2(x)ds \\
&= -Ae^{\alpha t}\int_0^t e^{\alpha s}T_{t-s}[T_s f]^2(x)\,ds. && (2.9)
\end{aligned}$$

Let $\mu_t := \mathbb{P}_\mu\langle f, X_t\rangle$. The moments are given by

$$\mathbb{P}_\mu(\langle f, X_t\rangle)^n = (-1)^n(e^{-\langle u_f,\mu\rangle})^{(n)}|_{\theta=0}.$$

In particular,

$$\mu_t=\mathbb{P}_\mu\langle f, X_t\rangle = \langle u_f^{(1)}(x,t,0),\mu\rangle = e^{\alpha t}\langle T_t f,\mu\rangle, \tag{2.10}$$

$$\mathbb{P}_\mu(\langle f, X_t\rangle-\mu_t)^2 = -\langle u_f^{(2)}(x,t,0),\mu\rangle. \tag{2.11}$$

Recall that $\widetilde{X}_t$ is defined in Section 2.1. It is a subcritical superprocess with branching mechanism $\psi^*(\lambda)=\psi(\lambda+\lambda^*)$. Thus $(\psi^*)^{(m)}(0+)=\psi^{(m)}(\lambda^*)$ exists for all $m\geq 1$. For any $f\in\mathcal{P}$, we define

$$u_f^*(x,t,\theta) = -\log\mathbb{P}_{\delta_x}(e^{-\langle\theta f,\widetilde{X}_t\rangle}). \tag{2.12}$$

Then

$$u_f^*(x,t,\theta)+\Pi_x\int_0^t \psi^*(u_f^*(\xi_s,t-s,\theta))ds=\theta\Pi_x f(\xi_t). \tag{2.13}$$

Differentiating both sides of (2.12) with respect to θ, we have

$$\begin{aligned}
(u_f^*)^{(1)}(x,t,0) &= e^{-\alpha^* t}T_t f(x), && (2.14)\\
(u_f^*)^{(2)}(x,t,0) &= -(\psi^*)^{(2)}(0+)\int_0^t e^{-(\psi^*)'(0+)(t-s)}T_{t-s}[(u_f^*)^{(1)}(\cdot,s,0)]^2(x)ds \\
&= -(\psi^*)^{(2)}(0+)e^{-\alpha^* t}\int_0^t e^{-\alpha^* s}T_{t-s}[T_s f]^2(x)\,ds, && (2.15)\\
(u_f^*)^{(3)}(x,t,0) &= -(\psi^*)^{(3)}(0+)\int_0^t e^{-\alpha^* s}T_s[(u_f^*)^{(1)}(\cdot,t-s,0)]^3(x)\,ds \\
&\quad -3(\psi^*)^{(2)}(0+)\int_0^t e^{-\alpha^* s}T_s[((u_f^*)^{(1)}(u_f^*)^{(2)})(\cdot,t-s,0)](x)\,ds, && (2.16)
\end{aligned}$$

and

$$(u_f^*)^{(4)}(x,t,0) = -\int_0^t e^{-\alpha^* s}T_s[J(\cdot,t-s)](x)\,ds, \tag{2.17}$$

where

$$J(x,t) = \left[(\psi^*)^{(4)}(0)\left((u_f^*)^{(1)}\right)^4 + 6(\psi^*)^{(3)}(0)\left((u_f^*)^{(1)}\right)^2 (u_f^*)^{(2)}\right](x,t,0) + \left[4(\psi^*)^{(2)}(0)(u_f^*)^{(1)}(u_f^*)^{(3)} + 3(\psi^*)^{(2)}(0)\left((u_f^*)^{(2)}\right)^2\right](x,t,0).$$

By (2.12), the moments of $\widetilde{X}$ are given by

$$\mathbb{P}_\mu(\langle f, \widetilde{X}_t\rangle)^n = (-1)^n (e^{-\langle u_f^*, \mu\rangle})^{(n)}|_{\theta=0}.$$

In particular, we have

$$\mathbb{P}_\mu\langle f, \widetilde{X}_t\rangle = \langle (u_f^*)^{(1)}(x,t,0), \mu\rangle = e^{-\alpha^* t}\langle T_t f, \mu\rangle, \tag{2.18}$$

$$\mathbb{P}_\mu(\langle f, \widetilde{X}_t\rangle - \mathbb{P}_\mu\langle f, \widetilde{X}_t\rangle)^2 = -\langle (u_f^*)^{(2)}(x,t,0), \mu\rangle. \tag{2.19}$$

$$\mathbb{P}_\mu(\langle f, \widetilde{X}_t\rangle - \mathbb{P}_\mu\langle f, \widetilde{X}_t\rangle)^4 = -\langle (u_f^*)^{(4)}(x,t,0), \mu\rangle + 3\langle (u_f^*)^{(2)}(x,t,0), \mu\rangle^2. \tag{2.20}$$

2.3 Estimates on the semigroup T_t

Recall that $\xi = \{\xi_t : t \ge 0\}$ is the OU process and $\{T_t\}$ is the semigroup of ξ. It is well known that under Π_x, $\xi_t \sim \mathcal{N}(xe^{-bt}, \sigma_t^2)$, where $\sigma_t^2 = \sigma^2(1-e^{-2bt})/(2b)$. Let G be an $\mathbb{R}^d$-valued standard normal random variable, then using $(a+b)^n \le 2^n(a^n + b^n),\ a \ge 0,\ b \ge 0$, we get

$$T_t(\|\cdot\|^n)(x) = E(\|\sigma_t G + xe^{-bt}\|^n) \le 2^n \left[(\sigma/\sqrt{2b})^n E(\|G\|^n) + \|x\|^n\right]. \tag{2.21}$$

Using this, we can easily get that

$$T_t(1 + \|\cdot\|^n)(x) \le c(n)(1 + \|x\|^n), \tag{2.22}$$

where $c(n)$ does not depend on t.

Lemma 2.2 *For any $f \in L^2(\varphi)$, we have that, for every $x \in \mathbb{R}^d$,*

$$T_t f(x) = \sum_{n=\gamma(f)}^{\infty} e^{-nbt} \sum_{|p|=n} a_p \phi_p(x), \tag{2.23}$$

$$\lim_{t\to\infty} e^{\gamma(f)bt} T_t f(x) = \sum_{|p|=\gamma(f)} a_p \phi_p(x). \tag{2.24}$$

Moreover, there exists $c > 0$ such that for $t \ge 1$,

$$|T_t f(x)| \le c e^{-\gamma(f)bt} e^{\frac{b}{2\sigma^2}\|x\|^2}, \quad x \in \mathbb{R}^d. \tag{2.25}$$

Proof: For every $f \in L^2(\varphi)$, using the fact that $\varphi(x)$ is the invariant density of ξ we get that

$$\int \varphi(x) \left(T_t|f|(x)\right)^2 \, dx \le \int \varphi(x) T_t[|f|^2](x) \, dx = \int |f(y)|^2 \varphi(y) \, dy < \infty, \tag{2.26}$$

so $T_t f(x) \in L^2(\varphi)$. Moreover, by the fact $\xi_t \sim \mathcal{N}(xe^{-bt}, \sigma_t^2)$, $T_t|f|(x)$ is continuous in x. Thus $T_t|f|(x) < \infty$ for all $x \in \mathbb{R}^d$. (2.26) implies that T_t is a bounded linear operator on $L^2(\varphi)$. Let $f_k(x) = \sum_{n=0}^k \sum_{|p|=n} a_p \phi_p(x)$. Since $f_k \to f$ in $L^2(\varphi)$, we have $T_t f_k \to T_t f$ in $L^2(\varphi)$, as $k \to \infty$. By linearity, we have

$$T_t f_k(x) = \sum_{n=0}^{k} e^{-nbt} \left(\sum_{|p|=n} a_p \phi_p(x) \right).$$

We claim that the series $\sum_{n=0}^\infty e^{-nbt} \left(\sum_{|p|=n} a_p \phi_p(x) \right)$ is uniformly convergent on any compact subset of $\mathbb{R}^d$. Thus $\sum_{n=0}^\infty e^{-nbt} \left(\sum_{|p|=n} a_p \phi_p(x) \right)$ is continuous in x. So for all $x \in \mathbb{R}^d$,

$$T_t f(x) = \sum_{n=0}^{\infty} e^{-nbt} \left(\sum_{|p|=n} a_p \phi_p(x) \right).$$

Now we prove the claim. In fact, by Cramer's inequality (for example, see [15, Equation (19) on p.207]), for all $p \in \mathbb{Z}_+^d$ we have

$$|\phi_p(x)| \le K e^{\frac{b}{2\sigma^2}\|x\|^2}, \tag{2.27}$$

where K is a constant. So we only need to prove $\sum_{n=0}^\infty e^{-nbt} \left(\sum_{|p|=n} |a_p| \right) < \infty$. By Hölder's inequality,

$$\sum_{n=0}^{\infty} e^{-nbt} \left(\sum_{|p|=n} |a_p| \right) \le \left(\sum_{n=\gamma(f)}^{\infty} K_n e^{-2nbt} \right)^{1/2} \left(\sum_{n=\gamma(f)}^{\infty} \sum_{|p|=n} |a_p|^2 \right)^{1/2}, \tag{2.28}$$

where $K_n = \binom{n+d-1}{d-1} = \sharp\{p \in \mathbb{Z}_+^d : |p| = n\}$. Since $K_n \le (n+d)^d$, we have that $\sum_{n=\gamma(f)}^\infty K_n e^{-2nbt} < \infty$. Using the fact that $\{\phi_p(x),\ p \in \mathbb{Z}_+^d\}$ form a complete orthogonal basis for $L^2(\varphi)$, we get $\sum_{n=\gamma(f)}^\infty \sum_{|p|=n} |a_p|^2 = \int \varphi(x)|f(x)|^2 \, dx < \infty$. Therefore the claim is true.

By (2.27) and (2.28), for $t \ge 1$, we have

$$\begin{aligned} |T_t f(x)| &\le e^{-\gamma(f)bt} \left(\sum_{n=0}^{\infty} K_{n+\gamma(f)} e^{-2nb} \right)^{1/2} \Big(\sum_{n=\gamma(f)}^{\infty} \sum_{|p|=n} |a_p|^2 \Big)^{1/2} K e^{\frac{b}{2\sigma^2}\|x\|^2} \\ &\lesssim e^{-\gamma(f)bt} e^{\frac{b}{2\sigma^2}\|x\|^2}, \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.29}$$

Therefore, for $t \ge 1$,

$$|e^{\gamma(f)bt} T_t f(x) - \sum_{|p|=\gamma(f)} a_p \phi_p(x)| = e^{\gamma(f)bt} |T_t f(x) - e^{-\gamma(f)bt} \sum_{|p|=\gamma(f)} a_p \phi_p(x)|$$

$$
\begin{aligned}
&= e^{\gamma(f)bt}\left|T_t(f-\sum_{|p|=\gamma(f)} a_p\phi_p)(x)\right| \\
&\lesssim e^{-bt}e^{\frac{b}{2\sigma^2}\|x\|^2}, \qquad (2.30)
\end{aligned}
$$

which implies (2.24). The proof is now complete. □

For $p \in \mathbb{Z}_+^d$, we use the notation $f^{(p)}(x) := \frac{\partial}{\partial x_1^{p_1}\partial x_2^{p_2}\cdots\partial x_d^{p_d}} f(x)$. Define

$$
\mathcal{P}^* = \{f \in C^\infty : \ f^{(p)} \in \mathcal{P} \text{ for all } p \in \mathbb{Z}_+^d\}.
$$

It can be easily shown that, for any $f \in \mathcal{P}$, $T_t f(x) \in \mathcal{P}^*$.

Lemma 2.3 *For any $f \in \mathcal{P}^*$ and $p \in \mathbb{Z}_+^d$ satisfying $0 \le |p| \le \gamma(f)$, we have $\gamma(f^{(p)}) \ge \gamma(f) - |p|$.*

Proof: By the definition of ϕ_p and φ, it is easy to check that

$$
\phi_p(x)\varphi(x) = (-1)^{|p|}c_p\varphi^{(p)}(x),
$$

where $c_p = \frac{1}{\sqrt{p!2^{|p|}}}\left(\frac{\sigma^2}{b}\right)^{|p|/2}$. Integrating by parts, we get

$$
\int f(x)\phi_p(x)\varphi(x)\,dx = c_p\int_{\mathbb{R}^d} f^{(p)}(x)\varphi(x)\,dx. \qquad (2.31)
$$

Thus

$$
\gamma(f^{(p)}) = \inf\{k : \text{ there exists } p \text{ such that } |p| = k \text{ and } \int_{\mathbb{R}^d} f^{(p)}(x)\varphi(x)\,dx \neq 0\}.
$$

Hence if $|p'| < \gamma(f) - |p|$, we have $\int_{\mathbb{R}^d} f^{(p+p')}(x)\varphi(x)\,dx = 0$, which implies $\gamma(f^{(p)}) \ge \gamma(f) - |p|$. □

In the following lemma, we give another estimate for $T_t f$, which will be very useful later.

Lemma 2.4 *For every $f \in \mathcal{P}$, there exist $r \in \mathbb{N}$ and $c > 0$ such that*

$$
e^{\gamma(f)bt}|T_t f(x)| \le c(1 + \|x\|^r), \qquad (2.32)
$$

$$
\left|e^{\gamma(f)bt}T_t f(x) - \sum_{|p|=\gamma(f)} a_p\phi_p(x)\right| \le ce^{-bt}(1 + \|x\|^r). \qquad (2.33)
$$

Proof: Let $g(x) = T_1 f(x) \in \mathcal{P}^*$. Then $\gamma(g) = \gamma(f)$ and there exist $k \in \mathbb{N}$ and $c_1 > 0$ such that, for $|p| = 0, 1\cdots, \gamma(f)$, $|g^{(p)}(x)| \le c_1(1+\|x\|^k)$. For $x = (x_1, x_2\cdots, x_d) \in \mathbb{R}^d$, we define $x^p := \prod_{i=1}^d x_i^{p_i}$. Then for $s > 0$ we have

$$
\begin{aligned}
T_s g(x) &= T_s[g(\cdot + xe^{-bs})](0) \\
&= T_s\left[g(\cdot + xe^{-bs}) - \sum_{m=0}^{\gamma(f)-1}\sum_{|p|=m} g^{(p)}(\cdot)x^p e^{-mbs}/p!\right](0)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{m=0}^{\gamma(f)-1} \sum_{|p|=m} T_s[g^{(p)}](0) x^p e^{-mbs}/p! \\
&= \quad (I) + (II).
\end{aligned}
$$

It follows from (2.25) and the fact that $\gamma(g^{(p)}) \ge \gamma(g) - |p|$, we have

$$
\sup_{s>0} e^{(\gamma(g)-|p|)bs} |T_s[g^{(p)}](0)| < \infty.
$$

Thus

$$
|(II)| \lesssim e^{-\gamma(f)bs} \sum_{m=0}^{\gamma(f)-1} \sum_{|p|=m} |x^p| \lesssim e^{-\gamma(f)bs}(1 + \|x\|^{\gamma(f)}).
$$

Using Taylor's formula and the fact $|g^{(p)}(x)| \lesssim 1 + \|x\|^k$, we get

$$
\begin{aligned}
\left| g(y + xe^{-bs}) - \sum_{m=0}^{\gamma(f)-1} \sum_{|p|=m} g^{(p)}(y) x^p e^{-mbs}/p! \right| &= \sum_{|p|=\gamma(f)} |g^{(p)}(\theta)||x^p| e^{-\gamma(f)bs}/(\gamma(f)!) \\
&\lesssim (1 + \|y\|^k + \|x\|^k)|x|^{\gamma(f)} e^{-\gamma(f)bs},
\end{aligned}
$$

where θ is a point on the line segment connecting y and $y+xe^{-bs}$. Then by the fact that $T_s[\|\cdot\|^k](x) \lesssim 1 + \|x\|^k$, we get $\sup_{s>0} T_s[\|\cdot\|^k](0) < \infty$. Therefore, we have

$$
|(I)| \lesssim (1 + \|x\|^{k+\gamma(f)}) e^{-\gamma(f)bs}.
$$

Consequently,

$$
e^{\gamma(f)bs} |T_s g|(x) \lesssim 1 + \|x\|^{k+\gamma(f)}.
$$

Let $r_1 = k + \gamma(f)$. For $t \ge 1$, combining $T_t f(x) = T_{t-1}(g)(x)$ with the above inequality, we arrive at (2.32) for $t \ge 1$. For $t < 1$,

$$
e^{\gamma(f)bt} |T_t f(x)| \lesssim e^{\gamma(f)b}(1 + \|x\|^k) \lesssim 1 + \|x\|^{r_1},
$$

so (2.32) is also valid.

It follows from (2.32) that there exists $r_2 \in \mathbb{N}$ such that

$$
e^{(\gamma(f)+1)bt} \left| T_t f(x) - e^{-\gamma(f)bt} \sum_{|p|=\gamma(f)} a_p \phi_p(x) \right| \lesssim 1 + \|x\|^{r_2}.
$$

Now (2.33) follows immediately. $\square$

From the above calculations, we have

Lemma 2.5 *Let $f \in \mathcal{P}$.*

(i) If $\alpha < 2\gamma(f)b$, then

$$\lim_{t\to\infty} e^{-(\alpha/2)t}\mathbb{P}_{\delta_x}(\langle f, X_t\rangle) = 0,$$

$$\lim_{t\to\infty} e^{-\alpha t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle = \sigma_f^2, \tag{2.34}$$

where $\mathbb{V}ar_{\delta_x}$ stands for the variance under $\mathbb{P}_{\delta_x}$ and σ_f^2 is defined in (1.14).

(ii) If $\alpha = 2\gamma(f)b$, then

$$\lim_{t\to\infty} t^{-1/2}e^{-(\alpha/2)t}\mathbb{P}_{\delta_x}(\langle f, X_t\rangle) = 0, \tag{2.35}$$

and there exists $r \in \mathbb{N}$ such that

$$|t^{-1}e^{-\alpha t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle| \lesssim 1 + \|x\|^{2r} \tag{2.36}$$

and

$$|t^{-1}e^{-\alpha t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle - \rho_f^2| \lesssim t^{-1}(1 + \|x\|^r), \tag{2.37}$$

which in particular implies that

$$\lim_{t\to\infty} t^{-1}e^{-\alpha t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle = \rho_f^2, \tag{2.38}$$

where ρ_f^2 is defined in (1.17).

(iii) If $\alpha > 2\gamma(f)b$, then

$$\lim_{t\to\infty} e^{-2(\alpha-\gamma(f)b)t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle = \eta_f^2(x), \tag{2.39}$$

where

$$\eta_f^2(x) = A\int_0^\infty e^{-(\alpha-2\gamma(f)b)s}T_s\left(\sum_{|p|=\gamma(f)} a_p\phi_p\right)^2(x)\,ds. \tag{2.40}$$

Proof: It follows from (2.9) and (2.11) that

$$\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle = Ae^{\alpha t}\int_0^t e^{\alpha s}T_{t-s}[T_s f]^2(x)ds = Ae^{2\alpha t}\int_0^t e^{-\alpha s}T_s[T_{t-s}f]^2(x)ds. \tag{2.41}$$

(i) If $\alpha < 2\gamma(f)b$, by Lemma 2.2, we have $\lim_{t\to\infty} e^{\gamma(f)bt}T_t f(x) = \sum_{|p|=\gamma(f)} a_p\phi_p(x)$. Thus

$$\lim_{t\to\infty} e^{-(\alpha/2)t}\mathbb{P}_{\delta_x}\langle f, X_t\rangle = \lim_{t\to\infty} e^{(\alpha-2\gamma(f)b)t/2}[e^{\gamma(f)bt}T_t f(x)] = 0.$$

It follows from Lemma 2.4 that there exists $r \in \mathbb{N}$ such that $e^{\gamma(f)bs}|T_s f|(x) \lesssim 1 + \|x\|^r$. Using (2.22), we have

$$T_{t-s}[e^{\gamma(f)bs}T_s f]^2(x) \lesssim 1 + \|x\|^{2r}. \tag{2.42}$$

Thus $e^{\alpha s}T_{t-s}[T_s f]^2(x) \lesssim e^{(\alpha-2\gamma(f)b)s}(1+\|x\|^{2r})$. Hence by the dominated convergence theorem, we get

$$\lim_{t\to\infty}\int_0^t e^{\alpha s}T_{t-s}[T_s f]^2(x)\,ds = \int_0^\infty e^{\alpha s}\langle (T_s f)^2, \varphi\rangle\,ds,$$

which implies (2.34).

(ii) If $\alpha = 2\gamma(f)b$, then by (2.41), we have

$$t^{-1}e^{-\alpha t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle = At^{-1}\int_0^t T_{t-s}[e^{\gamma(f)bs}T_s f]^2(x)\,ds. \tag{2.43}$$

By Lemma 2.4, there exists $r \in \mathbb{N}$ satisfying (2.32), (2.33) and

$$|\sum_{|p|=\gamma(f)} a_p\phi_p(x)| \lesssim 1 + \|x\|^r,$$

which follows from the fact that $\phi_p(x)$ is a polynomial. Then by (2.32) and (2.43), it is easy to get (2.36).

Let $h(x) := (\sum_{|p|=\gamma(f)} a_p\phi_p(x))^2$. Then we have

$$\begin{aligned} & |(e^{\gamma(f)bs}T_s f(x))^2 - h(x)| \\ \leq\ & \left|e^{\gamma(f)bs}T_s f(x) - \sum_{|p|=\gamma(f)} a_p\phi_p(x)\right| \left(e^{\gamma(f)bs}|T_s f|(x) + |\sum_{|p|=\gamma(f)} a_p\phi_p(x)|\right) \\ \lesssim\ & e^{-bs}(1 + \|x\|^{2r}). \end{aligned}$$

Since $\gamma(h) = 0$ and $\sum_{|p|=\gamma(f)} a_p^2 = \langle h, \varphi\rangle$, by (2.33), there exists $r' \in \mathbb{N}$ such that

$$\left|T_{t-s}h(x) - \sum_{|p|=\gamma(f)} a_p^2\right| \lesssim e^{-b(t-s)}(1 + \|x\|^{r'}). \tag{2.44}$$

Let $r_0 = \max(2r, r')$, then

$$\begin{aligned} \left|T_{t-s}(e^{\gamma(f)bs}T_s f)^2(x) - \sum_{|p|=\gamma(f)} a_p^2\right| &\leq T_{t-s}|(e^{\gamma(f)bs}T_s f(x))^2 - h(x)| + \left|T_{t-s}h(x) - \sum_{|p|=\gamma(f)} a_p^2\right| \\ &\lesssim (e^{-bs} + e^{-b(t-s)})(1 + \|x\|^{r_0}). \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{1}{t}\int_0^t |T_{t-s}(e^{\gamma(f)bs}T_s f)^2(x) - \sum_{|p|=\gamma(f)} a_p^2|\,ds \\ \lesssim\ & \frac{\int_0^t (e^{-bs} + e^{-b(t-s)})(1 + \|x\|^{r_0})\,ds}{t} \lesssim t^{-1}(1 + \|x\|^{r_0}). \end{aligned} \tag{2.45}$$

Then (2.37) follows from (2.43) and (2.45).

(iii) If $\alpha > 2\gamma(f)b$, then by (2.41), we have

$$e^{-2(\alpha-\gamma(f)b)t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle = A\int_0^t e^{-(\alpha-2\gamma(f)b)s}T_s[e^{\gamma(f)b(t-s)}T_{t-s}f]^2(x)\,ds.$$

By Lemma 2.4, there exists $r \in \mathbb{N}$ such that $[e^{\gamma(f)b(t-s)}T_{t-s}f(x)]^2 \le c(1+\|x\|^{2r})$. Thus

$$T_s[e^{\gamma(f)b(t-s)}T_{t-s}f]^2(x) \lesssim 1+\|x\|^{2r}.$$

Now by the dominated convergence theorem and (2.24), we have

$$\lim_{t\to\infty}\int_0^t e^{-(\alpha-2\gamma(f)b)s}T_s[e^{\gamma(f)b(t-s)}T_{t-s}f]^2(x)\,ds = A\int_0^\infty e^{-(\alpha-2\gamma(f)b)s}T_s\left(\sum_{|p|=\gamma(f)} a_p\phi_p\right)^2(x)\,ds.$$

The proof of (iii) is now complete. □

According to [8], under $\mathbf{P}_{\delta_x}$, we have that, conditioned on $\mathcal{F}_t$ (see (2.5)), the backbone Z_t is a Poisson point process with the intensity $\lambda^*\Lambda_t$. In particular, $Z_0 = N\delta_x$, where N is a Poisson random variable with parameter λ^*. Then we have

$$\Lambda_t = \widetilde{X}_t + \sum_{j=1}^N I_t^j, \tag{2.46}$$

where $I^j, j = 1, 2, ...$ are independent copies of I under $\mathbb{Q}_{\delta_x}$ and are independent of N. The first moment of I can be calculated by

$$\mathbf{P}_{\delta_x}\langle f, \Lambda_t\rangle = \mathbf{P}_{\delta_x}\langle f, \widetilde{X}_t\rangle + \lambda^*\mathbb{Q}_{\delta_x}\langle f, I_t\rangle. \tag{2.47}$$

Thus

$$\nu_t := \mathbb{Q}_{\delta_x}\langle f, I_t\rangle = \frac{1}{\lambda^*}\left(\mathbf{P}_{\delta_x}\langle f, \Lambda_t\rangle - \mathbf{P}_{\delta_x}\langle f, \widetilde{X}_t\rangle\right) = \frac{1}{\lambda^*}(e^{\alpha t} - e^{-\alpha^* t})T_t f(x). \tag{2.48}$$

For the second moment, let $\mathbf{V}ar_{\delta_x}$ stand for the variance under $\mathbf{P}_{\delta_x}$ and $\mathbb{V}_{\delta_x}$ stand for the variance under $\mathbb{Q}_{\delta_x}$. By (2.46), we have

$$\mathbf{V}ar_{\delta_x}\langle f, \Lambda_t\rangle = \mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_t\rangle + \lambda^*\mathbb{Q}_{\delta_x}\langle f, I_t\rangle^2.$$

Thus

$$\mathbb{Q}_{\delta_x}\langle f, I_t\rangle^2 = \frac{1}{\lambda^*}(\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle - \mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_t\rangle). \tag{2.49}$$

Corollary 2.6 *Let $\{I_t\}_{t\ge 0}$ be the process described in the Subsection 2.1 and $f \in \mathcal{P}$.*

(i) If $\alpha < 2\gamma(f)b$, then

$$\lim_{t\to\infty} e^{-(\alpha/2)t}\mathbb{Q}_{\delta_x}(\langle f, I_t\rangle) = 0, \tag{2.50}$$

$$\lim_{t\to\infty} e^{-\alpha t}\mathbb{V}_{\delta_x}\langle f, I_t\rangle = \frac{A}{\lambda^*}\int_0^\infty e^{\alpha s}\langle (T_s f)^2, \varphi\rangle\,ds = \frac{\sigma_f^2}{\lambda^*}. \tag{2.51}$$

(ii) If $\alpha = 2\gamma(f)b$, then

$$\lim_{t\to\infty} t^{-1/2}e^{-(\alpha/2)t}\mathbb{Q}_{\delta_x}\left(\langle f, I_t\rangle\right) = 0, \tag{2.52}$$

and there exists $r \in \mathbb{N}$ such that

$$|t^{-1}e^{-\alpha t}\mathbb{V}_{\delta_x}\langle f, I_t\rangle| \lesssim 1 + \|x\|^{2r} \tag{2.53}$$

and

$$\left|t^{-1}e^{-\alpha t}\mathbb{V}_{\delta_x}\langle f, I_t\rangle - \frac{A}{\lambda^*}\sum_{|p|=\gamma(f)} a_p^2\right| \lesssim t^{-1}(1 + \|x\|^r), \tag{2.54}$$

which in particular implies that

$$\lim_{t\to\infty} t^{-1}e^{-\alpha t}\mathbb{V}_{\delta_x}\langle f, I_t\rangle = \frac{A}{\lambda^*}\sum_{|p|=\gamma(f)} a_p^2. \tag{2.55}$$

(iii) If $\alpha > 2\gamma(f)b$, then

$$\lim_{t\to\infty} e^{-2(\alpha-\gamma(f)b)t}\mathbb{V}_{\delta_x}\langle f, I_t\rangle = \frac{\eta_f^2(x)}{\lambda^*} - \frac{1}{(\lambda^*)^2}\Big(\sum_{|p|=\gamma(f)} a_p\phi_p(x)\Big)^2. \tag{2.56}$$

Proof: Using (2.46) and Lemma 2.5, we can easily obtain the corollary. Here we just give the proof of (2.51). By (2.49), we have

$$e^{-\alpha t}\mathbb{V}_{\delta_x}\langle f, I_t\rangle = \frac{1}{\lambda^*}e^{-\alpha t}\mathbb{V}ar_{\delta_x}\langle f, X_t\rangle - \frac{1}{\lambda^*}e^{-\alpha t}\mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_t\rangle - e^{-\alpha t}(\mathbb{Q}_{\delta_x}\langle f, I_t\rangle)^2. \tag{2.57}$$

Using (2.15) and (2.19), we have

$$\mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_t\rangle = (\psi^*)''(0+)e^{-\alpha^* t}\int_0^t e^{-\alpha^* s}T_{t-s}[T_s f]^2(x)\,ds. \tag{2.58}$$

By the fact that there exists $r \in \mathbb{N}$ such that $|T_t f(x)| \lesssim 1+\|x\|^r$, we get $T_{t-s}[T_s f]^2(x) \lesssim (1+\|x\|^{2r})$. Thus

$$\mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_t\rangle \lesssim e^{-\alpha^* t}(1 + \|x\|^{2r}) \to 0, \quad t \to \infty. \tag{2.59}$$

By (2.48), $|\mathbb{Q}_{\delta_x}\langle f, I_t\rangle| \lesssim e^{\alpha t}|T_t f(x)| \lesssim e^{(\alpha-\gamma(f)b)t}(1 + \|x\|^r)$, thus we have

$$\lim_{t\to\infty} e^{-\alpha t}(\mathbb{Q}_{\delta_x}\langle f, I_t\rangle)^2 \lesssim \lim_{t\to\infty} e^{(\alpha-2\gamma(f)b)t}(1 + \|x\|^{2r}) = 0. \tag{2.60}$$

Now, using (2.34), (2.59) and (2.60), we easily get (2.51). □

Lemma 2.7 *For $f \in \mathcal{P}$, it holds that*

$$\mathbb{P}_\mu(\langle f, \widetilde{X}_t\rangle - \mathbb{P}_\mu\langle f, \widetilde{X}_t\rangle)^4 \lesssim \langle 1 + \|x\|^{4r}, \mu\rangle + \langle 1 + \|x\|^{2r}, \mu\rangle^2. \tag{2.61}$$

Proof: By (2.32), there exists $r \in \mathbb{N}$ such that $|T_t f(x)| \lesssim 1+\|x\|^r$. So by (2.14), $|(u_f^*)^{(1)}(x,t,0)| \lesssim 1 + \|x\|^r$. By (2.59) and (2.19), we have $|(u_f^*)^{(2)}(x,t,0)| \lesssim 1 + \|x\|^{2r}$. Thus using (2.16), we get $|(u_f^*)^{(3)}(x,t,0)| \lesssim 1 + \|x\|^{3r}$. Then by (2.17), we have $|(u_f^*)^{(4)}(x,t,0)| \lesssim 1 + \|x\|^{4r}$. Now (2.61) follows immediately from (2.20). □

3 Proofs of the main theorems

In this section, we will prove the main results of this paper. Recall that we assume that the initial measure μ is a finite measure on $\mathbb{R}^d$ with compact support, and that $(X_t, \mathbb{P}_\mu)$ and $(\Lambda_t, \mathbf{P}_\mu)$ have the same law. Thus in the remainder of this paper, we will replace $(X_t, \mathbb{P}_\mu)$ by $(\Lambda_t, \mathbf{P}_\mu)$. Define

$$\mathcal{L}_t = \{u \in \mathcal{T}, \tau_u \le t < \sigma_u\}, \quad t \ge 0.$$

From the construction of Λ_t, we have

$$\Lambda_{(t+s)} = \widetilde{X}^t_s + \sum_{u \in \mathcal{L}_t} I^{u,t}_s, \tag{3.1}$$

where, conditioned on $\mathcal{G}_t$, $\widetilde{X}^t$ is a superprocess with the same law as X under $\mathbb{P}^*_{\Lambda_t}$ and $I^{u,t}$ has the same law as I under $\mathbb{Q}_{z_u(t)}$. The processes $I^{u,t}, u \in \mathcal{L}_t$, are independent.

3.1 The large rate case: $\alpha > 2b\gamma(f)$

Recall that

$$H^p_t = e^{-(\alpha-|p|b)t}\langle \phi_p, X_t\rangle, \quad t \ge 0.$$

Lemma 3.1 *H^p_t is a martingale under $\mathbb{P}_\mu$. Moreover, if $\alpha > 2|p|b$, we have $\sup_t \mathbb{P}_\mu(H^p_t)^2 < \infty$, and therefore the limit*

$$H^p_\infty := \lim_{t\to\infty} H^p_t$$

exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$.

Proof: Since ϕ_p is an eigenfunction of L corresponding to $-|p|b$, by (2.10), we have $\mathbb{P}_\mu H^p_t = \langle \phi_p, \mu\rangle$. Thus, by the Markov property, we get that H^p_t is a martingale. Using (2.10) and (2.11), we get

$$\mathbb{P}_\mu\langle \phi_p, X_t\rangle^2 = e^{2(\alpha-|p|b)t}\langle \phi_p, \mu\rangle^2 + Ae^{\alpha t}\int_{\mathbb{R}^d}\int_0^t e^{(\alpha-2|p|b)s}T_{t-s}[\phi_p^2](x)\,ds\,\mu(dx).$$

Thus when $\alpha > 2|p|b$, we have by the definition of H^p_t,

$$\begin{aligned} \mathbb{P}_\mu(H^p_t)^2 &= \langle \phi_p, \mu\rangle^2 + A\int_{\mathbb{R}^d}\int_0^t e^{-(\alpha-2|p|b)s}T_s[\phi_p^2](x)\,ds\,\mu(dx) \\ &\le \langle \phi_p, \mu\rangle^2 + A\int_{\mathbb{R}^d}\int_0^\infty e^{-(\alpha-2|p|b)s}T_s[\phi_p^2](x)\,ds\,\mu(dx). \end{aligned}$$

Since $|\phi_p^2| \lesssim 1 + \|x\|^{2|p|}$, by (2.22), we have $|T_s[\phi_p^2](x)| \lesssim 1 + \|x\|^{2|p|}$. Thus

$$\int_{\mathbb{R}^d}\int_0^\infty e^{-(\alpha-2|p|b)s}T_s[\phi_p^2](x)\,ds\,\mu(dx) \lesssim \int_{\mathbb{R}^d}(1+\|x\|^{2|p|})\,\mu(dx) < \infty, \tag{3.2}$$

from which the convergence asserted in the lemma follow easily. $\square$

We now present the proof of Theorem 1.1.

Proof of Theorem 1.1: Define $M_t := e^{-(\alpha-\gamma(f)b)t)}\langle \widetilde{f}, X_t\rangle$, where

$$\widetilde{f}(x) = f(x) - \sum_{|p|=\gamma(f)} a_p\phi_p(x) = \sum_{n=\gamma(f)+1}^{\infty} \sum_{|p|=n} a_p\phi_p(x).$$

It is clear that $\gamma(\widetilde{f}) \geq \gamma(f)+1$. From Lemma 2.5 and (2.32), we have

(1) If $\alpha > 2\gamma(\widetilde{f}))b$, then

$$\lim_{t\to\infty} e^{-2(\alpha-\gamma(\widetilde{f})b)t)}\mathbb{P}_\mu\langle \widetilde{f}, X_t\rangle^2 \tag{3.3}$$

exists, thus we have

$$\begin{aligned}\mathbb{P}_\mu M_t^2 &= e^{-2(\gamma(\widetilde{f})-\gamma(f))bt}e^{-2(\alpha-\gamma(\widetilde{f})b)t)}\mathbb{P}_\mu\langle \widetilde{f}, X_t\rangle^2 \\ &= O(e^{-2(\gamma(\widetilde{f})-\gamma(f))bt}) \to 0, \quad \text{ as } t\to\infty.\end{aligned}$$

(2) If $\alpha = 2\gamma(\widetilde{f})b$, then $\lim_{t\to\infty} t^{-1}e^{-\alpha t}\mathbb{P}_\mu\langle \widetilde{f}, X_t\rangle^2$ exists. Thus we have

$$\begin{aligned}\mathbb{P}_\mu M_t^2 &= te^{-2(\gamma(\widetilde{f})-\gamma(f))t}(t^{-1}e^{-\alpha t}\mathbb{P}_\mu\langle \widetilde{f}, X_t\rangle^2) \\ &= O(te^{-2(\gamma(\widetilde{f})-\gamma(f))t}) \to 0, \quad \text{ as } t\to\infty.\end{aligned}$$

(3) If $2\gamma(f)b < \alpha < 2\gamma(\widetilde{f})b$, then $\lim_{t\to\infty} e^{-\alpha t}\mathbb{P}_\mu\langle \widetilde{f}, X_t\rangle^2$ exists. Thus we have

$$\begin{aligned}\mathbb{P}_\mu M_t^2 &= e^{-(\alpha-2\gamma(f)b)t}(e^{-\alpha t}\mathbb{P}_\mu\langle \widetilde{f}, X_t\rangle^2) \\ &= O(e^{-(\alpha-2\gamma(f)b)t}) \to 0, \quad \text{ as } t\to\infty.\end{aligned}$$

Combining the three cases above, we get $\lim_{t\to\infty} M_t = 0$ in $L^2(\mathbb{P}_\mu)$. Now using Lemma 3.1, we easily get the convergence in Theorem 1.1. $\square$

3.2 The small rate case: $\alpha < 2\gamma(f)b$

First, we recall some property of weak convergence. For $f : \mathbb{R}^d \to \mathbb{R}$, let $\|f\|_L := \sup_{x\neq y} |f(x) - f(y)|/\|x-y\|$ and $\|f\|_{BL} := \|f\|_\infty + \|f\|_L$. For any distributions ν_1 and ν_2 on $\mathbb{R}^d$, define

$$\beta(\nu_1,\nu_2) := \sup\left\{\left|\int f\,d\nu_1 - \int f\,d\nu_2\right| \ :\ \|f\|_{BL} \leq 1\right\}.$$

Then β is a metric. By [9, Theorem 11.3.3], the topology generated by this metric is equivalent to the weak convergence topology. From the definition, we can easily see that, if ν_1 and ν_2 are the distributions of two $\mathbb{R}^d$-valued random variables X and Y respectively, then

$$\beta(\nu_1,\nu_2) \leq E\|X-Y\| \leq \sqrt{E\|X-Y\|^2}. \tag{3.4}$$

In the following, we will use the following elementary fact: If X is a real-valued random variable with $E|X|^n < \infty$, then

$$\left| E(e^{i\theta X} - \sum_{m=0}^{n} \frac{(i\theta X)^m}{m!}) \right| \le \frac{|\theta|^n}{n!} E\left(|X|^n \left(\frac{|\theta X|}{n+1} \wedge 2 \right) \right), \tag{3.5}$$

which is an immediate consequence of the simple inequality

$$\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right| \le \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right).$$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3: We define an $\mathbb{R}^2$-valued random variable $U_1(t)$ by

$$U_1(t) := (e^{-\alpha t} \|\Lambda_t\|, e^{-(\alpha/2)t} \langle f, \Lambda_t \rangle). \tag{3.6}$$

To get the conclusion of Theorem 1.3, it suffices to show that under $\mathbf{P}_\mu$,

$$U_1(t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty} G_1(f)), \tag{3.7}$$

where $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Let $s, t > 0$ and write

$$U_1(s+t) = (e^{-\alpha(s+t)} \|\Lambda_{s+t}\|, e^{-(\alpha/2)(s+t)} \langle f, \Lambda_{s+t} \rangle).$$

Recall the representation (3.1). Define

$$Y_s^{u,t} := e^{-\alpha s/2} \langle f, I_s^{u,t} \rangle \quad \text{and} \quad y_s^{u,t} := \mathbf{P}_\mu(Y_s^{u,t} | \mathcal{G}_t). \tag{3.8}$$

$Y_s^{u,t}$ has the same law as $Y_s := e^{-\alpha s/2} \langle f, I_s \rangle$ under $\mathbb{Q}_{\delta_{Z_u(t)}}$. Then we have

$$\begin{aligned}
& e^{-(\alpha/2)(s+t)} \langle f, \Lambda_{s+t} \rangle \\
= \;& e^{-(\alpha/2)(s+t)} \langle f, \widetilde{X}_s^t \rangle + e^{-(\alpha/2)t} \sum_{u \in \mathcal{L}_t} Y_s^{u,t} \\
= \;& e^{-(\alpha/2)(s+t)} (\langle f, \widetilde{X}_s^t \rangle - \mathbf{P}_\mu(\langle f, \widetilde{X}_s^t \rangle | \mathcal{G}_t) \\
& + e^{-(\alpha/2)t} \sum_{u \in \mathcal{L}_t} (Y_s^{u,t} - y_s^{u,t}) + e^{-(\alpha/2)(t+s)} \mathbf{P}_\mu(\langle f, \Lambda_{s+t} \rangle | \mathcal{G}_t) \\
=: \;& J_0(s,t) + J_1(s,t) + J_2(s,t).
\end{aligned} \tag{3.9}$$

Put $\widetilde{V}_s(x) := \mathbf{V}ar_{\delta_x} \langle f, \widetilde{X}_s \rangle$. Then

$$\mathbf{P}_\mu J_0(s,t)^2 = e^{-\alpha(t+s)} \mathbf{P}_\mu \langle \widetilde{V}_s, \Lambda_t \rangle = e^{-\alpha s} \langle T_t \widetilde{V}_s, \mu \rangle.$$

By (2.59), there exists $r \in \mathbb{N}$ such that $\widetilde{V}_s(x) \lesssim e^{-\alpha^* s}(1 + \|x\|^{2r})$. Thus

$$\mathbf{P}_\mu J_0(s,t)^2 \lesssim e^{-\alpha s} e^{-\alpha^* s} \int_{\mathbb{R}^d} (1 + \|x\|^{2r}) \mu(dx). \tag{3.10}$$

Next we consider $J_2(s,t)$. By the Markov property and (2.10), we have

$$J_2(s,t) = e^{-(\alpha/2)(s+t)} e^{\alpha s} \langle T_s f, \Lambda_t \rangle.$$

Thus, by (2.9) and (2.10), we have

$$\begin{aligned}\mathbf{P}_\mu J_2(s,t)^2 &= Ae^{\alpha s} \int_{\mathbb{R}^d} \int_0^t e^{\alpha u} T_{t-u}[T_{s+u} f]^2(x)\, du\, \mu(dx) + e^{\alpha(t+s)} \langle T_{t+s} f, \mu \rangle^2 \\ &\lesssim e^{(\alpha - 2\gamma(f)b)s} \int_{\mathbb{R}^d} (1 + \|x\|^{2r})\, \mu(dx), \end{aligned} \tag{3.11}$$

here the last inequality follows from the fact that there exists $r \in \mathbb{N}$ such that

$$|T_{s+u} f|(x) \lesssim e^{-\gamma(f)b(u+s)}(1 + \|x\|^r).$$

Thus by (3.10) and (3.11), we have

$$\lim_{s\to\infty} \limsup_{t\to\infty} \mathbf{P}_\mu (J_0(s,t) + J_2(s,t))^2 = 0. \tag{3.12}$$

Now we consider $J_1(s,t)$. We define an $\mathbb{R}^2$-valued random variable $U_2(s,t)$ by

$$U_2(s,t) := \left(e^{-\alpha t}\|\Lambda_t\|, J_1(s,t)\right).$$

We claim that under $\mathbf{P}_\mu$,

$$U_2(s,t) \stackrel{d}{\to} (W_\infty, \sqrt{W_\infty} G_1(s)), \qquad \text{as } t \to \infty, \tag{3.13}$$

where $G_1(s) \sim \mathcal{N}(0, \sigma_f^2(s))$ and $\sigma_f^2(s)$ will be given later. Denote the characteristic function of $U_2(s,t)$ under $\mathbf{P}_\mu$ by $\kappa(\theta_1, \theta_2, s, t)$:

$$\begin{aligned}\kappa(\theta_1,\theta_2,s,t) &= \mathbf{P}_\mu\left(\exp\left\{i\theta_1 e^{-\alpha t}\|\Lambda_t\| + i\theta_2 e^{-(\alpha/2)t} \sum_{u\in\mathcal{L}_t} (Y_s^{u,t} - y_s^{u,t})\right\}\right) \\ &= \mathbf{P}_\mu\left(\exp\{i\theta_1 e^{-\alpha t}\|\Lambda_t\|\} \prod_{u\in\mathcal{L}_t} h_s(z_u(t), e^{-(\alpha/2)t}\theta_2)\right) \\ &= \mathbf{P}_\mu\left(\exp\{i\theta_1 e^{-\alpha t}\|\Lambda_t\|\} \exp\left\{\lambda^* \langle h_s(\cdot,\ e^{-(\alpha/2)t}\theta_2) - 1, \Lambda_t\rangle\right\}\right), \end{aligned} \tag{3.14}$$

where $h_s(x,\theta) = \mathbb{Q}_{\delta_x} e^{i\theta(Y_s - \mathbb{Q}_{\delta_x} Y_s)}$. The last equality in the display above follows from the fact that given Λ_t, Z_t is a Poisson random measure with intensity $\lambda^*\Lambda_t$. Define

$$e_s(x,\theta) := h_s(x,\theta) - 1 + \frac{1}{2}\theta^2 \mathbb{V}_{\delta_x} Y_s$$

and $V_s(x) := \mathbb{V}_{\delta_x} Y_s$. Then

$$\exp\left\{\lambda^*\langle h_s(\cdot,\ e^{-(\alpha/2)t}\theta_2) - 1, \Lambda_t\rangle\right\} = \exp\left\{-\lambda^* \frac{1}{2}\theta_2^2 e^{-\alpha t}\langle V_s, \Lambda_t\rangle\right\} \exp\left\{\lambda^*\langle e_s(\cdot, e^{-(\alpha/2)t}\theta_2), \Lambda_t\rangle\right\}$$

$$= \quad J_{1,1}(s,t)J_{1,2}(s,t).$$

By (3.5), we have

$$|e_s(x, e^{-(\alpha/2)t}\theta_2)| \le \theta_2^2 e^{-\alpha t}\mathbb{Q}_{\delta_x}\left(|Y_s - \mathbb{Q}_{\delta_x}Y_s|^2(\frac{e^{-(\alpha/2)t}\theta_2|Y_s - \mathbb{Q}_{\delta_x}Y_s|}{6}\wedge 1)\right).$$

Let

$$g(x,s,t) := \mathbb{Q}_{\delta_x}\left(|Y_s - \mathbb{Q}_{\delta_x}Y_s|^2(\frac{e^{-(\alpha/2)t}\theta_2|Y_s - \mathbb{Q}_{\delta_x}Y_s|}{6}\wedge 1)\right).$$

We notice $g(x,s,t)\downarrow 0$ as $t\uparrow\infty$. By (2.10),

$$\mathbf{P}_\mu|\langle e_s(\cdot, e^{-(\alpha/2)t}\theta_3), \Lambda_t\rangle| \le \theta_2^2\ \langle\ T_t(g(\cdot,s,t)),\ \mu\ \rangle \to 0, \quad \text{as} \quad t\to\infty.$$

Thus $\lim_{t\to\infty}\langle e_s(\cdot, e^{-(\alpha/2)t}\theta_2), \Lambda_t\rangle = 0$ in probability, which implies $\lim_{t\to\infty} J_{1,2}(s,t) = 1$ in probability. Furthermore, by Remark 1.2, we have

$$\lim_{t\to\infty} e^{-\alpha t}\langle V_s, \Lambda_t\rangle = \langle V_s,\ \varphi\rangle W_\infty \quad \text{in probability,}$$

which implies that $\lim_{t\to\infty} J_{1,1}(s,t) = \exp\left\{-\frac{1}{2}\theta_2^2\sigma_f^2(s)W_\infty\right\}$, where $\sigma_f^2(s) := \lambda^*\langle V_s,\ \varphi\rangle$. Thus

$$\lim_{t\to\infty}\exp\left\{\lambda^*\langle h_s(\cdot,\ e^{-(\alpha/2)t}\theta_2) - 1, \Lambda_t\rangle\right\} = \exp\left\{-\frac{1}{2}\theta_2^2\sigma_f^2(s)W_\infty\right\} \quad \text{in probability.} \tag{3.15}$$

Since $h_s(x,\theta)$ is a characteristic function, its real part is less than 1, which implies

$$\left|\exp\left\{\lambda^*\langle h_s(\cdot,\ e^{-(\alpha/2)t}\theta_2) - 1, \Lambda_t\rangle\right\}\right| \le 1.$$

Hence by the dominated convergence theorem, we get

$$\lim_{t\to\infty}\kappa(\theta_1,\theta_2,s,t) = \mathbf{P}_\mu\exp\{i\theta_1 W_\infty\}\exp\left\{-\frac{1}{2}\theta_2^2\sigma_f^2(s)W_\infty\right\}, \tag{3.16}$$

which implies our claim (3.13). Thus, we easily get that under $\mathbf{P}_\mu$,

$$U_3(s,t) := \left(e^{-\alpha(t+s)}\|\Lambda_{t+s}\|, J_1(s,t)\right) \xrightarrow{d} (W_\infty, \sqrt{W_\infty}G_1(s)), \quad \text{as } t\to\infty.$$

By (2.51), we have $\lim_{s\to\infty} V_s(x) = \frac{\sigma_f^2}{\lambda^*}$, thus $\lim_{s\to\infty}\sigma_f^2(s) = \sigma_f^2$. So

$$\lim_{s\to\infty}\beta(G_1(s), G_1(f)) = 0. \tag{3.17}$$

Let $\mathcal{L}(s+t)$ and $\widetilde{\mathcal{L}}(s,t)$ be the distributions of $U_1(s+t)$ and $U_3(s,t)$ respectively, and let $\mathcal{L}(s)$ and $\mathcal{L}$ be the distributions of $(W_\infty, \sqrt{W_\infty}G_1(s))$ and $(W_\infty, \sqrt{W_\infty}G_1(f))$ respectively. Then, using (3.4), we have

$$\limsup_{t\to\infty}\beta(\mathcal{L}(s+t),\mathcal{L}) \quad \le \quad \limsup_{t\to\infty}[\beta(\mathcal{L}(s+t),\widetilde{\mathcal{L}}(s,t)) + \beta(\widetilde{\mathcal{L}}(s,t),\mathcal{L}(s)) + \beta(\mathcal{L}(s),\mathcal{L})]$$

$$\leq \limsup_{t\to\infty}(\mathbf{P}_\mu(J_0(s,t)+J_2(s,t))^2)^{1/2}+0+\beta(\mathcal{L}(s),\mathcal{L}). \tag{3.18}$$

Using this and the definition of $\limsup_{t\to\infty}$, we easily get that

$$\limsup_{t\to\infty}\beta(\mathcal{L}(t),\mathcal{L})=\limsup_{t\to\infty}\beta(\mathcal{L}(s+t),\mathcal{L})\leq \limsup_{t\to\infty}(\mathbf{P}_\mu(J_0(s,t)+J_2(s,t))^2)^{1/2}+\beta(\mathcal{L}(s),\mathcal{L}).$$

Letting $s\to\infty$, we get $\limsup_{t\to\infty}\beta(\mathcal{L}(t),\mathcal{L})=0$. The proof is now complete. □

3.3 Proof of Theorem 1.9

In this section we consider the case $\alpha>2\gamma(f)b$ and $f_{(c)}=0$. Recall the decomposition of Λ_t under $\mathbf{P}_{\delta_x}$ in (2.46), we have for $|p|=m<\alpha/(2b)$,

$$H^p_s=e^{-(\alpha-mb)s}\langle\phi_p,\widetilde{X}_s\rangle+\sum_{j=1}^N e^{-(\alpha-mb)s}\langle\phi_p,I^j_s\rangle. \tag{3.19}$$

Let

$$\widetilde{H}^p_s:=e^{-(\alpha-mb)(s)}\langle\phi_p,I_s\rangle.$$

Then under $\mathbf{P}_{\delta_x}$, the processes $\{e^{-(\alpha-mb)s}\langle\phi_p,I^j_s\rangle,s\geq 0\}$, $j=1,2\dots$ are i.i.d. with a common law equal to that of $\{\widetilde{H}^p_s,s\geq 0\}$ under $\mathbb{Q}_{\delta_x}$. Since ϕ_p is an eigenvalue of L corresponding to $-|p|b$, we have

$$\mathbf{P}_{\delta_x}\langle\phi_p,\widetilde{X}_s\rangle=e^{-(\alpha^*+mb)s}\phi_p(x)\to 0,\quad \text{as } s\to\infty. \tag{3.20}$$

Thus, by (2.59), we have that as $s\to\infty$,

$$\mathbf{P}_{\delta_x}(\langle\phi_p,\widetilde{X}_s\rangle)^2\lesssim e^{-\alpha^* s}(1+\|x\|^{2|p|})\to 0, \tag{3.21}$$

which implies $\lim_{s\to\infty}e^{-(\alpha-mb)s}\langle\phi_p,\widetilde{X}_s\rangle=0$ in $L^2(\mathbf{P}_{\delta_x})$. By Lemma 3.1, $\lim_{s\to\infty}H^p_s=H^p_\infty$ in $L^2(\mathbf{P}_{\delta_x})$. Thus

$$\lim_{s\to\infty}\sum_{j=1}^N e^{-(\alpha-mb)s}\langle\phi_p,I^j_s\rangle=H^p_\infty\quad \text{in } L^2(\mathbf{P}_{\delta_x}). \tag{3.22}$$

From the fact that N is independent of I^j, we have for any $s,t\geq 0$,

$$\begin{aligned}&\mathbf{P}_{\delta_x}\left[\sum_{j=1}^N(e^{-(\alpha-mb)s}\langle\phi_p,I^j_s\rangle-e^{-(\alpha-mb)t}\langle\phi_p,I^j_t\rangle)\right]^2\\ \geq\ &\mathbf{P}_{\delta_x}[(e^{-(\alpha-mb)s}\langle\phi_p,I_s\rangle-e^{-(\alpha-mb)t}\langle\phi_p,I_t\rangle)^2;N=1]\\ =\ &\mathbf{P}_{\delta_x}(N=1)\mathbb{Q}_{\delta_x}(\widetilde{H}^p_s-\widetilde{H}^p_t)^2.\end{aligned}$$

By (3.22), we get for any $x\in\mathbb{R}^d$,

$$\mathbb{Q}_{\delta_x}(\widetilde{H}^p_s-\widetilde{H}^p_t)^2\to 0,\quad s,t\to\infty. \tag{3.23}$$

Thus $\widetilde{H}^p_s$ converges in $L^2(\mathbb{Q}_{\delta_x})$. Let

$$\widetilde{H}^p_\infty := \lim_{s\to\infty} \widetilde{H}^p_s \quad \text{in } L^2(\mathbb{Q}_{\delta_x}),$$

which implies $H^{j,p}_\infty := \lim_{s\to\infty}\langle \phi_p, I^j_s\rangle e^{-(\alpha-mb)s}$ exists in $L^2(\mathbf{P}_{\delta_x})$. Furthermore, $H^{j,p}_\infty$, under $\mathbf{P}_{\delta_x}$, are i.i.d. with a common law equal to that of $\widetilde{H}^p_\infty$ under $\mathbb{Q}_{\delta_x}$. Hence by (3.22), it is easy to get

$$H^p_\infty = \sum_{j=1}^N H^{j,p}_\infty, \quad \mathbf{P}_{\delta_x}\text{- a.s.} \tag{3.24}$$

Recall the decomposition of Λ_{t+s} in (3.1). By Lemma 3.1, we have for $|p| = m < \alpha/(2b)$,

$$H^p_{t+s} = e^{-(\alpha-mb)(s+t)}\langle \phi_p, \widetilde{X}^t_s\rangle + e^{-(\alpha-mb)t}\sum_{u\in\mathcal{L}_t} e^{-(\alpha-mb)s}\langle \phi_p, I^{u,t}_s\rangle. \tag{3.25}$$

From the definition of $\widetilde{X}^t_s$, using (2.59) and (3.20), we have

$$\begin{aligned}
\mathbf{P}_\mu(\langle \phi_p, \widetilde{X}^t_s\rangle)^2 &\le 2\mathbf{P}_\mu(\langle \phi_p, \widetilde{X}^t_s\rangle - \mathbf{P}_\mu(\langle \phi_p, \widetilde{X}^t_s\rangle|\mathcal{F}_t))^2 + 2\mathbf{P}_\mu\left(\mathbf{P}_\mu(\langle \phi_p, \widetilde{X}^t_s\rangle|\mathcal{F}_t)\right)^2\\
&= 2\mathbf{P}_\mu\langle \mathbf{V}ar_{\delta_\cdot}\langle \phi_p, \widetilde{X}_s\rangle, \Lambda_t\rangle + 2\mathbf{P}_\mu\langle \mathbf{P}_{\delta_\cdot}\langle \phi_p, \widetilde{X}_s\rangle, \Lambda_t\rangle^2 \to 0, \quad \text{as } s\to\infty.
\end{aligned}$$

Hence $\lim_{s\to\infty} e^{-(\alpha-mb)(s+t)}\langle \phi_p, \widetilde{X}^t_s\rangle = 0$ in $L^2(\mathbf{P}_\mu)$. Thus $\lim_{s\to\infty} e^{-(\alpha-mb)(s+t)}\langle \phi_p, \widetilde{X}^t_s\rangle = 0$ in $L^2(\mathbf{P}_\mu)$. Thus

$$\lim_{s\to\infty} e^{-(\alpha-mb)t}\sum_{u\in\mathcal{L}_t}\langle \phi_p, I^{u,t}_s\rangle e^{-(\alpha-mb)s} = H^p_\infty \quad \text{in } L^2(\mathbf{P}_\mu). \tag{3.26}$$

Note that under $\mathbf{P}_\mu$, given Z_t, $e^{-(\alpha-mb)(s)}\langle \phi_p, I^{u,t}_s\rangle$ has the same law as $\widetilde{H}^p_s$ under $\mathbb{Q}_{\delta_{Z_u(t)}}$. Thus by (3.23), for each $u\in\mathcal{L}_t$, $e^{-(\alpha-mb)s}\langle \phi_p, I^{u,t}_s\rangle$ converges in $L^2(\mathbf{P}_\mu)$ to a limit, denoted as $H^{u,t,p}_\infty$. Furthermore, given Z_t, $H^{u,t,p}_\infty$ has the same law as $\widetilde{H}^p_\infty$ under $\mathbb{Q}_{\delta_{Z_u(t)}}$.

We claim that, for each $t\ge 0$,

$$H^p_\infty = e^{-(\alpha-mb)t}\sum_{u\in\mathcal{L}_t} H^{u,t,p}_\infty. \tag{3.27}$$

In fact,

$$\begin{aligned}
\mathbf{P}_\mu\Big(\sum_{u\in\mathcal{L}_t} e^{-(\alpha-mb)s}\langle \phi_p, I^{u,t}_s\rangle - H^{u,t,p}_\infty\Big)^2 &\le \mathbf{P}_\mu|Z_t|\sum_{u\in\mathcal{L}_t}(e^{-(\alpha-mb)s}\langle \phi_p, I^{u,t}_s\rangle - H^{u,t,p}_\infty)^2\\
&= \mathbf{P}_\mu|Z_t|\sum_{u\in\mathcal{L}_t}\mathbb{Q}_{\delta_{Z_u(t)}}(\widetilde{H}^p_s - \widetilde{H}^p_\infty)^2.
\end{aligned}$$

By (2.49), we have

$$\mathbb{Q}_{\delta_x}(\widetilde{H}^p_s)^2 \le \frac{1}{\lambda^*}\mathbb{V}ar_{\delta_x}(H^p_s) \le \frac{1}{\lambda^*}\mathbb{P}_{\delta_x}(H^p_s)^2 \lesssim 1 + \|x\|^{2|p|}.$$

Thus $\mathbb{Q}_{\delta_x}(\widetilde{H}^p_s - \widetilde{H}^p_\infty)^2 \le 2\sup_{s\ge 0}\mathbb{Q}_{\delta_x}(\widetilde{H}^p_s)^2 \lesssim 1+\|x\|^{2|p|}$. We can easily get that

$$\mathbf{P}_\mu |Z_t| \langle (1+\|\cdot\|^{2|p|}), Z_t\rangle < \infty.$$

So by the dominated convergence theorem, we have $\lim_{s\to\infty}\mathbf{P}_\mu(\sum_{u\in\mathcal{L}_t} e^{-(\alpha-mb)s}\langle\phi_p, I^{u,t}_s\rangle - H^{u,t,p}_\infty)^2 = 0$. Now the claim (3.27) follows easily from (3.26).

Define

$$H^{u,t}_\infty := \sum_{\gamma(f)\le m<\alpha/2b}\sum_{|p|=m} a_p H^{u,t,p}_\infty \quad \text{and} \quad \widetilde{H}_\infty := \sum_{\gamma(f)\le m<\alpha/2b}\sum_{|p|=m} a_p \widetilde{H}^p_\infty.$$

Recall the definition of H_∞ in (1.19). By (3.24), we have

$$H_\infty = \sum_{u\in\mathcal{L}_0} H^{u,0}_\infty.$$

Under $\mathbf{P}_{\delta_x}$, $H^{u,0}_\infty$ are i.i.d. with a common law equal to that of $\widetilde{H}_\infty$ under $\mathbb{Q}_{\delta_x}$. Thus we have

$$\mathbf{P}_{\delta_x} H_\infty = \lambda^* \mathbb{Q}_{\delta_x}\widetilde{H}_\infty, \tag{3.28}$$

$$\mathbf{V}ar_{\delta_x} H_\infty = \lambda^* \mathbb{Q}_{\delta_x}(\widetilde{H}_\infty)^2. \tag{3.29}$$

On the other hand, by Lemma 3.1, we get

$$\lim_{t\to\infty}\sum_{\gamma(f)\le m<\alpha/2b} e^{-(\alpha-mb)t}\sum_{|p|=m} a_p\langle\phi_p,\Lambda_t\rangle = H_\infty, \quad \text{in } L^2(\mathbf{P}_{\delta_x}).$$

It follows that

$$\mathbf{P}_{\delta_x} H_\infty = f_{(s)}(x), \tag{3.30}$$

and by (2.41),

$$\mathbf{V}ar_{\delta_x} H_\infty = A\int_0^\infty e^{-\alpha s} T_s\left(\sum_{\gamma(f)\le m<\alpha/2b} e^{mbs}\sum_{|p|=m} a_p\phi_p\right)^2(x)\,ds. \tag{3.31}$$

Proof of Theorem 1.9: By (3.27), we have

$$\sum_{\gamma(f)\le m<\alpha/2b} e^{(\alpha-mb)t}\sum_{|p|=m} a_p H^p_\infty = \sum_{u\in\mathcal{L}_t} H^{u,t}_\infty.$$

Consider the $\mathbb{R}^2$-valued random variable $U_1(t)$:

$$U_1(t) := \left(e^{-\alpha t}\|\Lambda_t\|,\ e^{-(\alpha/2)t}(\langle f,\Lambda_t\rangle - \sum_{u\in\mathcal{L}_t} H^{u,t}_\infty)\right). \tag{3.32}$$

To get the conclusion of Theorem 1.9, it suffices to show that

$$U_1(t) \stackrel{d}{\to} (W_\infty, \sqrt{W_\infty} G_3(f)). \tag{3.33}$$

Denote the characteristic function of $U_1(t)$ with respect to $\mathbf{P}_\mu$ by $\kappa_1(\theta_1, \theta_2, t)$ and let $h(x, \theta) := \mathbb{Q}_{\delta_x} \exp\{i\theta \widetilde{H}_\infty\}$. Then we have

$$\begin{aligned}
& \kappa_1(\theta_1, \theta_2, t) \\
= \;& \mathbf{P}_\mu \exp\left\{ i\theta_1 e^{-\alpha t} \|\Lambda_t\| + i\theta_2 e^{-(\alpha/2)t} (\langle f, \Lambda_t\rangle - \sum_{u \in \mathcal{L}_t} H_\infty^{u,t}) \right\} \\
= \;& \mathbf{P}_\mu \exp\left\{ i\theta_1 e^{-\alpha t} \|\Lambda_t\| \right\} \exp\left\{ i\theta_2 e^{-(\alpha/2)t} \langle f, \Lambda_t\rangle \right\} \prod_{u \in \mathcal{L}_t} h\left(Z_u(t), -\theta_2 e^{-(\alpha/2)t}\right) \\
= \;& \mathbf{P}_\mu \exp\left\{ i\theta_1 e^{-\alpha t} \|\Lambda_t\| \right\} \exp\left\{ i\theta_2 e^{-(\alpha/2)t} \langle f, \Lambda_t\rangle + \lambda^* \langle h(\cdot, -\theta_2 e^{-(\alpha/2)t}) - 1, \Lambda_t\rangle \right\}.
\end{aligned} \tag{3.34}$$

The third equality above follows from the fact that, given Λ_t, Z_t is a Poisson point process with density $\lambda^* \Lambda_t$. By (3.28) and (3.30), we get $\mathbb{Q}_{\delta_x} \widetilde{H}_\infty = f_{(s)}(x)/\lambda^*$. Let

$$e(x, \theta) := h(x, \theta) - 1 - \frac{i\theta}{\lambda^*} f_{(s)}(x) + \frac{1}{2} \mathbb{Q}_{\delta_x} (\widetilde{H}_\infty)^2 \theta^2$$

and $V(x) := \mathbf{V}ar_{\delta_x} H_\infty$. Then, by (3.29), we have

$$\begin{aligned}
& i\theta_2 e^{-(\alpha/2)t} \langle f, \Lambda_t\rangle + \lambda^* \langle h(\cdot, -\theta_2 e^{-(\alpha/2)t}) - 1, \Lambda_t\rangle \\
= \;& i\theta_2 e^{-(\alpha/2)t} \langle f_{(l)}, \Lambda_t\rangle - \frac{1}{2} \theta_2^2 e^{-\alpha t} \langle V, \Lambda_t\rangle + \lambda^* \langle e(\cdot, -\theta_2 e^{-(\alpha/2)t}), \Lambda_t\rangle \\
=: \;& J_1(t) + J_2(t) + J_3(t).
\end{aligned}$$

By (3.5), we have

$$|e(x, \theta)| \le \theta^2 \mathbb{Q}_{\delta_x} \left(|\widetilde{H}_\infty|^2 \left(\frac{\theta |\widetilde{H}_\infty|}{6} \wedge 1 \right) \right), \tag{3.35}$$

which implies that

$$|J_3(t)| \;\; \le \;\; \theta_2^2 e^{-\alpha t} \langle g(\cdot,\ t), \Lambda_t\rangle,$$

where

$$g(x, t) := \mathbb{Q}_{\delta_x} \left(|\widetilde{H}_\infty|^2 \left(\frac{e^{-(\alpha/2)t} \theta_2 |\widetilde{H}_\infty|}{6} \wedge 1 \right) \right).$$

It is clear that $g(x, t) \downarrow 0$ as $t \uparrow \infty$. Thus

$$\mathbf{P}_\mu |J_3(t)| \le \theta_2^2 \ \langle\ T_t(g(\cdot, t)),\ \mu\ \rangle \to 0, \quad \text{as} \quad t \to \infty, \tag{3.36}$$

which implies $\lim_{t\to\infty} J_3(t) = 0$ in probability. By Remark 1.2, we have

$$\lim_{t\to\infty} e^{-\alpha t} \langle V, \Lambda_t\rangle = \langle V,\ \varphi\rangle W_\infty \quad \text{in probability.} \tag{3.37}$$

Recall that $\lim_{t\to\infty} e^{-\alpha t}\|\Lambda_t\| = W_\infty$, $\mathbf{P}_\mu$-a.s. Therefore

$$\lim_{t\to\infty} \exp\left\{i\theta_1 e^{-\alpha t}\|\Lambda_t\|\right\}\exp\{J_2(t)+J_3(t)\} = \exp\{i\theta_1 W_\infty\}\exp\{-\frac{1}{2}\theta_2^2\langle V,\ \varphi\rangle W_\infty\} \quad \text{in probability.} \tag{3.38}$$

Thus by the dominated convergence theorem, we get that as $t\to\infty$,

$$\left|\kappa_1(\theta_1,\theta_2,t) - \mathbf{P}_\mu \exp\left\{i\theta_2 e^{-(\alpha/2)t}\langle f_{(l)},\Lambda_t\rangle\right\}\exp\{i\theta_1 W_\infty\}\exp\{-\frac{1}{2}\theta_2^2\langle V,\ \varphi\rangle W_\infty\}\right| \to 0. \tag{3.39}$$

Since $\alpha < 2\gamma(f_{(l)})b$, by Theorem 1.3, we have that as $t\to\infty$,

$$(e^{-\alpha t}\|\Lambda_t\|, e^{-(\alpha/2)t}\langle f_{(l)},\Lambda_t\rangle) \xrightarrow{d} (W_\infty, \sqrt{W_\infty}G_1(f_{(l)})), \tag{3.40}$$

where $G_1(f_{(l)}) \sim \mathcal{N}(0,\sigma^2_{f_{(l)}})$. Therefore,

$$\begin{aligned} &\lim_{t\to\infty}\mathbf{P}_\mu \exp\left\{i\theta_2 e^{-(\alpha/2)t}\langle f_{(l)},\Lambda_t\rangle\right\} e^{i\theta_1 W_\infty}\exp\{-\frac{1}{2}\theta_2^2\langle V,\ \varphi\rangle W_\infty\} \\ =\ & \mathbf{P}_\mu e^{i\theta_1 W_\infty}\exp\{-\frac{1}{2}\theta_2^2(\sigma^2_{f_{(l)}} + \langle V,\ \varphi\rangle)W_\infty\}. \end{aligned} \tag{3.41}$$

By (3.31), we get

$$\langle V,\ \varphi\rangle = A\sum_{\gamma(f)\le m<\alpha/2b}\frac{1}{\alpha-2mb}\sum_{|p|=m}a_p^2.$$

The proof is now complete. □

3.4 The critical case: $\alpha = 2\gamma(f)b$

To prove Theorem 1.5, we need the following lemma. The idea of the proof is mainly from [6].

Lemma 3.2 *Assume $f\in\mathcal{P}$ satisfies $\alpha = 2\gamma(f)b$. Define $T_t^\alpha f(x) := e^{\alpha t}T_t f(x) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle$ and*

$$S_t f := t^{-1/2}e^{-(\alpha/2)t}(\langle f, X_t\rangle - T_t^\alpha f(x)).$$

Then for any $c>0$ and $\delta>0$, we have

$$\lim_{t\to\infty}\mathbb{P}_{\delta_x}\left(|S_t f|^2; |S_t f| > ce^{\delta t}\right) = 0. \tag{3.42}$$

Proof: We write $t = [t]+\epsilon_t$, where $[t]$ is the integer part of t. Then

$$\begin{aligned} S_t f &= t^{-1/2}e^{-(\alpha/2)t}\left(\langle f, X_t\rangle - \langle T^\alpha_{\epsilon_t}f, X_{[t]}\rangle\right) + t^{-1/2}e^{-(\alpha/2)t}\left(\langle T^\alpha_{\epsilon_t}f, X_{[t]}\rangle - T^\alpha_t f(x)\right) \\ &= t^{-1/2}e^{-(\alpha/2)t}\left(\langle f, X_t\rangle - \langle T^\alpha_{\epsilon_t}f, X_{[t]}\rangle\right) + \left(\frac{[t]}{t}\right)^{1/2} e^{-\alpha\epsilon_t/2}S_{[t]}(T^\alpha_{\epsilon_t}f). \end{aligned} \tag{3.43}$$

Thus

$$\mathbb{P}_{\delta_x}\left(|S_t f|^2; |S_t f| > ce^{\delta t}\right)$$

$$
\begin{aligned}
&\le 2t^{-1}e^{-\alpha t}\mathbb{P}_{\delta_x}\left(|\langle f,X_t\rangle-\langle T^\alpha_{\epsilon_t}f,X_{[t]}\rangle|^2\right)+2\frac{[t]}{t}e^{-\alpha\epsilon_t}\mathbb{P}_{\delta_x}\left(|S_{[t]}(T^\alpha_{\epsilon_t}f)|^2;|S_tf|>ce^{\delta t}\right)\\
&\le 2t^{-1}e^{-\alpha t}\mathbb{P}_{\delta_x}\left(|\langle f,X_t\rangle-\langle T^\alpha_{\epsilon_t}f,X_{[t]}\rangle|^2\right)\\
&\quad +2\frac{[t]}{t}e^{-\alpha\epsilon_t}\mathbb{P}_{\delta_x}\left(|S_{[t]}(T^\alpha_{\epsilon_t}f)|^2;|S_{[t]}(T^\alpha_{\epsilon_t}f)|>ce^{\alpha\epsilon_t/2}e^{\delta[t]}\right)\\
&\quad +2\frac{[t]}{t}e^{-\alpha\epsilon_t}\mathbb{P}_{\delta_x}\left(|S_{[t]}(T^\alpha_{\epsilon_t}f)|^2;|S_{[t]}(T^\alpha_{\epsilon_t}f)|\le ce^{\alpha\epsilon_t/2}e^{\delta[t]},|S_tf|>ce^{\delta t}\right)\\
&=: A_1(t)+A_2(t)+A_3(t).
\end{aligned}
$$

To prove (3.42) we only need to prove that $\lim_{t\to\infty}A_j(t)=0$ for $j=1,2,3$. In the following we give the detailed proof of $\lim_{t\to\infty}A_2(t)=0$. The arguments for $A_1(t)$ and $A_3(t)$ are similar and are omitted. To prove $\lim_{t\to\infty}A_2(t)=0$ we only need to prove, for $m\in\mathbb{N}$,

$$
\lim_{m\to\infty}\sup_{0\le s<1}\mathbb{P}_{\delta_x}\left(|S_m(T^\alpha_sf)|^2;|S_m(T^\alpha_sf)|>ce^{\delta m}\right)=0. \tag{3.44}
$$

Let

$$
F(t,f,c,\delta):=\mathbb{P}_{\delta_x}\left(|S_tf|^2;|S_tf|>ce^{\delta t}\right).
$$

Then (3.44) is equivalent to

$$
\lim_{m\to\infty}\sup_{0\le s<1}F(m,T^\alpha_sf,c,\delta)=0. \tag{3.45}
$$

Note that

$$
\begin{aligned}
S_{m+1}(T^\alpha_sf) &= \left(\frac{1}{m+1}\right)^{1/2}e^{-(\alpha/2)(m+1)}\left(\langle T^\alpha_sf,X_{m+1}\rangle-\langle T^\alpha_{s+1}f,X_m\rangle\right)\\
&\quad +\left(\frac{1}{m+1}\right)^{1/2}e^{-(\alpha/2)(m+1)}\left(\langle T^\alpha_{s+1}f,X_m\rangle-T^\alpha_{m+s+1}f(x)\right)\\
&= \left(\frac{1}{m+1}\right)^{1/2}R(m,T^\alpha_sf)+\left(\frac{m}{m+1}\right)^{1/2}e^{-\alpha/2}S_m(T^\alpha_{s+1}f),
\end{aligned} \tag{3.46}
$$

where $R(t,f):=e^{-(\alpha/2)(t+1)}\left(\langle f,X_{t+1}\rangle-\langle T^\alpha_1f,X_t\rangle\right)$. Thus we have

$$
\begin{aligned}
&F(m+1,T^\alpha_sf,c,\delta)\\
&\le \mathbb{P}_{\delta_x}\left(|S_{m+1}(T^\alpha_sf)|^2;|S_m(T^\alpha_{s+1}f)|>ce^{\alpha/2}e^{\delta m}\right)\\
&\quad +\mathbb{P}_{\delta_x}\left(|S_{m+1}(T^\alpha_sf)|^2;|S_m(T^\alpha_{s+1}f)|\le ce^{\alpha/2}e^{\delta m},|S_{m+1}(T^\alpha_sf)|>ce^{\delta(m+1)}\right)\\
&=: M_1(m,T^\alpha_sf,c,\delta)+M_2(m,T^\alpha_sf,c,\delta).
\end{aligned}
$$

Put

$$
\begin{aligned}
A_1(m,f,c,\delta) &= \{|S_m(T^\alpha_1f)|>ce^{\alpha/2}e^{\delta m}\},\\
A_2(m,f,c,\delta) &= \{|S_m(T^\alpha_1f)|\le ce^{\alpha/2}e^{\delta m},|S_{m+1}f|>ce^{\delta(m+1)}\}.
\end{aligned}
$$

Since $A_1(m, T_s^\alpha f, c, \delta) \in \mathcal{F}_m$ and $\mathbb{P}_{\delta_x}(R(m, T_s^\alpha f)|\mathcal{F}_m)$=0, we have by (3.46) that

$$\begin{aligned} M_1(m, T_s^\alpha f, c, \delta) &= \frac{1}{m+1}\mathbb{P}_{\delta_x}\left(|R(m, T_s^\alpha f)|^2; A_1(m, T_s^\alpha f, c, \delta)\right) \\ &\quad + \frac{m}{m+1}e^{-\alpha}\mathbb{P}_{\delta_x}\left(|S_m(T_{s+1}^\alpha f)|^2; A_1(m, T_s^\alpha f, c, \delta)\right) \\ &= \frac{1}{m+1}\mathbb{P}_{\delta_x}\left(|R(m, T_s^\alpha f)|^2; A_1(m, T_s^\alpha f, c, \delta)\right) \\ &\quad + \frac{m}{m+1}e^{-\alpha}F\left(m, T_{s+1}^\alpha f, ce^{\alpha/2}, \delta\right), \end{aligned}$$

and

$$\begin{aligned} M_2(m, T_s^\alpha f, c, \delta) &\le \frac{2}{m+1}\mathbb{P}_{\delta_x}\left(|R(m, T_s^\alpha f)|^2; A_2(m, T_s^\alpha f, c, \delta)\right) \\ &\quad + \frac{2m}{m+1}e^{-\alpha}\mathbb{P}_{\delta_x}\left(|S_m(T_{s+1}^\alpha f)|^2; A_2(m, T_s^\alpha f, c, \delta)\right). \end{aligned}$$

Thus we have

$$\begin{aligned} F(m+1, T_s^\alpha f, c, \delta) &\le \frac{m}{m+1}e^{-\alpha}F\left(m, T_{s+1}^\alpha f, ce^{\alpha/2}, \delta\right) \\ &\quad + \frac{1}{m+1}(G_1(m, T_s^\alpha f, c, \delta) + G_2(m, T_s^\alpha f, c, \delta)), \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} G_1(m, T_s^\alpha f, c, \delta) &= 2\mathbb{P}_{\delta_x}\left(|R(m, T_s^\alpha f)|^2; A_1(m, T_s^\alpha f, c, \delta) \cup A_2(m, T_s^\alpha f, c, \delta)\right), \\ G_2(m, T_s^\alpha f, c, \delta) &= 2me^{-\alpha}\mathbb{P}_{\delta_x}\left(|S_m(T_{s+1}^\alpha f)|^2; A_2(m, T_s^\alpha f, c, \delta)\right). \end{aligned}$$

Iterating (3.47), we get

$$\begin{aligned} F(m+1, T_s^\alpha f, c, \delta) &\le \frac{1}{m+1}\sum_{k=0}^{m} e^{-k\alpha}G_1(m-k, T_{k+s}^\alpha f, ce^{\alpha k/2}, \delta) \\ &\quad + \frac{1}{m+1}\sum_{k=0}^{m} e^{-k\alpha}G_2(m-k, T_{k+s}^\alpha f, ce^{\alpha k/2}, \delta) \\ &=: L_1(s, f, m) + L_2(s, f, m). \end{aligned} \tag{3.48}$$

Therefore, to prove (3.45), we only need to prove that

$$\sup_{0\le s<1} L_1(s, f, m) \to 0 \quad \text{and} \quad \sup_{0\le s<1} L_2(s, f, m) \to 0, \quad \text{as } m \to \infty. \tag{3.49}$$

First, we consider $L_1(s, f, m)$. Let $\widetilde{f}(x) = f(x) - \sum_{|p|=\gamma(f)} a_p\phi_p(x)$. Denote

$$A_{k,m,s} = A_1(m-k, T_{k+s}^\alpha f, ce^{\alpha k/2}, \delta) \cup A_2(m-k, T_{k+s}^\alpha f), ce^{\alpha k/2}, \delta).$$

Then we have

$$\begin{aligned}
L_1(s,f,m) &= \frac{2}{m+1}\sum_{k=0}^{m} e^{-k\alpha}\mathbb{P}_{\delta_x}(|R(m-k,T^{\alpha}_{k+s}f)|^2;A_{k,m,s}) \\
&\le \frac{4}{m+1}\sum_{k=0}^{m} e^{-k\alpha}\mathbb{P}_{\delta_x}(|R(m-k,T^{\alpha}_{k+s}\widetilde{f})|^2) \\
&\quad +\frac{4}{m+1}\sum_{k=0}^{m} e^{-k\alpha}\mathbb{P}_{\delta_x}(|R(m-k,e^{\alpha(k+s)/2}\sum_{|p|=\gamma(f)}a_p\phi_p)|^2;A_{k,m,s}) \\
&\le \frac{4}{m+1}\sum_{k=0}^{m} e^{-k\alpha}\mathbb{P}_{\delta_x}(|R(m-k,T^{\alpha}_{k+s}\widetilde{f})|^2) \\
&\quad +\frac{4e^{\alpha}}{m+1}\sum_{k=0}^{m}\mathbb{P}_{\delta_x}(|R(k,\sum_{|p|=\gamma(f)}a_p\phi_p)|^2;A_{m-k,m,s}) \\
&=: L_{1,1}(s,f,m)+L_{1,2}(s,f,m).
\end{aligned}$$

For $L_{1,1}(s,f,m)$, by the Markov property, we have

$$\begin{aligned}
\mathbb{P}_{\delta_x}(|R(m-k,T^{\alpha}_{k+s}\widetilde{f})|^2 &= e^{-\alpha(m-k+1)}\mathbb{P}_{\delta_x}\left(\langle T^{\alpha}_{k+s}\widetilde{f},X_{m-k+1}\rangle-\langle T^{\alpha}_{k+s+1}\widetilde{f},X_{m-k}\rangle\right)^2 \\
&= e^{-\alpha(m-k+1)}\mathbb{P}_{\delta_x}\langle \mathbb{V}ar_{\delta_\cdot}\langle T^{\alpha}_{s+k}\widetilde{f},X_1\rangle,\ X_{m-k}\rangle \\
&= e^{-\alpha}T_{m-k}(\mathbb{V}ar_{\delta_\cdot}\langle T^{\alpha}_{k+s}\widetilde{f},X_1\rangle)(x).
\end{aligned} \tag{3.50}$$

By (2.32), there exists $r\in\mathbb{N}$ such that $|T^{\alpha}_{k+s}\widetilde{f}(x)|=e^{\alpha(k+s)}|T_{k+s}\widetilde{f}(x)|\lesssim e^{(\alpha/2)k}e^{-bk}(1+\|x\|^r)$ for $0\le s<1$. So by (2.41), we obtain

$$\mathbb{V}ar_{\delta_x}\langle T^{\alpha}_{k+s}\widetilde{f},X_1\rangle = Ae^{\alpha}\int_0^1 e^{\alpha u}T_{1-u}[T_uT^{\alpha}_{k+s}\widetilde{f}]^2(x)du\lesssim e^{\alpha k}e^{-2bk}(1+\|x\|^{2r}),\quad s\in[0,1). \tag{3.51}$$

Thus $\mathbb{P}_{\delta_x}(|R(m-k,T^{\alpha}_{k+s}\widetilde{f})|^2\lesssim e^{\alpha k}e^{-2bk}(1+\|x\|^{2r})$. So,

$$L_{1,1}(s,f,m)\lesssim \frac{1}{m+1}\sum_{k=0}^{\infty}e^{-2bk}(1+\|x\|^{2r})\lesssim\frac{1}{m+1}(1+\|x\|^{2r})\to 0\quad m\to\infty. \tag{3.52}$$

Now we consider $L_{1,2}(s,f,m)$. Using (3.9) with $t=k$, $s=1$ and the function f replaced by $f_1:=\sum_{|p|=\gamma(f)}a_p\phi_p(x)$, we have

$$\begin{aligned}
R(k,\sum_{|p|=\gamma(f)}a_p\phi_p) &= e^{-(\alpha/2)(k+1)}\left(\langle f_1,X_{k+1}\rangle-\langle T^{\alpha}_1 f_1,X_k\rangle\right) \\
&= e^{-(\alpha/2)(k+1)}(\langle f_1,\widetilde{X}^k_1\rangle-\mathbf{P}_{\delta_x}(\langle f_1,\widetilde{X}^k_1\rangle|\mathcal{G}_k)+e^{-(\alpha/2)k}\sum_{u\in\mathcal{L}_k}(Y^{u,k}_1-y^{u,k}_1) \\
&=: J_0(k)+J_1(k),
\end{aligned}$$

where $Y_t^{u,k}, y_t^{u,k}$ are defined in (3.8). So for any $\epsilon > 0$,

$$\begin{aligned} L_{1,2}(s,f,m) &\leq \frac{4e^\alpha}{m+1}\sum_{k<m\epsilon}\mathbb{P}_{\delta_x}(|R(k,\sum_{|p|=\gamma(f)}a_p\phi_p)|^2) + \frac{8e^\alpha}{m+1}\sum_{m\epsilon\leq k\leq m}\mathbb{P}_{\delta_x}(|J_0(k)|^2; A_{m-k,t,s}) \\ &\quad + \frac{8e^\alpha}{m+1}\sum_{m\epsilon\leq k\leq m}\mathbb{P}_{\delta_x}(|J_1(k)|^2; A_{m-k,t,s}) \\ &=: (I) + (II) + (III). \end{aligned} \tag{3.53}$$

Using arguments similar to those leading to (3.50) and (3.51), we get

$$\mathbb{P}_{\delta_x} R(k, \sum_{|p|=\gamma(f)} a_p\phi_p)^2 = e^{-\alpha} T_k(\mathbb{V}ar_{\delta_\cdot}\langle \sum_{|p|=\gamma(f)} a_p\phi_p, X_1\rangle)(x) \lesssim 1 + \|x\|^{2\gamma(f)}.$$

Thus

$$(I) \lesssim \epsilon(1 + \|x\|^{2\gamma(f)}). \tag{3.54}$$

For (II) and (III), we claim that

(i) $|J_0(k)|^2$ and $|J_1(k)|^2$,$k = 1, 2, ...$ are both uniformly integrable with respect to $\mathbb{P}_{\delta_x}$;

(ii) $\sup_{k>m\epsilon}\sup_{0\leq s<1}\mathbb{P}_{\delta_x}(A_{m-k,m,s}) \to 0$ as $m \to \infty$.

Using the claims, we have

$$\begin{aligned} &\sup_{0\leq s<1}\frac{1}{m+1}\sum_{m\epsilon\leq k\leq m}\mathbb{P}_{\delta_x}(|J_0(k)|^2; A_{m-k,m,s}) \\ \leq\ & \frac{1}{m+1}\sum_{m\epsilon\leq k\leq m}\mathbb{P}_{\delta_x}(|J_0(k)|^2; |J_0(k)| > M) \\ &+ \frac{1}{m+1}\sum_{m\epsilon\leq k\leq m}\sup_{0\leq s<1}\mathbb{P}_{\delta_x}(|J_0(k)|^2; |J_0(k)| \leq M, A_{m-k,m,s}) \\ \leq\ & \sup_{k\geq 1}\mathbb{P}_{\delta_x}(|J_0(k)|^2; |J_0(k)| > M) + M^2\sup_{k>m\epsilon}\sup_{0\leq s<1}\mathbb{P}_{\delta_x}(A_{m-k,m,s}). \end{aligned}$$

First letting $m \to \infty$ and then $M \to \infty$, we get

$$\sup_{0\leq s<1}\frac{1}{m+1}\sum_{m\epsilon\leq k\leq m}\mathbb{P}_{\delta_x}(|J_0(k)|^2; A_{m-k,m,s}) \to 0 \quad \text{as } m \to \infty.$$

Similarly, we also have

$$\sup_{0\leq s<1}\frac{1}{m+1}\sum_{m\epsilon\leq k\leq m}\mathbb{P}_{\delta_x}(|J_1(k)|^2; A_{m-k,m,s}) \to 0, \quad \text{as } m \to \infty.$$

Thus, we have

$$\limsup_{m\to\infty}\sup_{0\leq s<1} L_{1,2}(s,f,m) \lesssim \epsilon(1 + \|x\|^{2\gamma(f)}). \tag{3.55}$$

Letting $\epsilon \to 0$, we get $\lim_{m\to\infty} \sup_{0\le s<1} L_{1,2}(s,f,m) = 0$. Therefore, by (3.52), we get

$$\lim_{m\to\infty} \sup_{0\le s<1} L_1(s,f,m) = 0.$$

Now we prove the claims (i) and (ii).

(i) For $J_0(k)$, by (2.61), there exists $r \in \mathbb{N}$ such that

$$\begin{aligned}\mathbb{P}_{\delta_x}|J_0(k)|^4 &\le e^{-2\alpha(k+1)}\mathbb{P}_{\delta_x}\left(\left\langle 1+\|\cdot\|^{4r}, X_k\right\rangle + \left\langle 1+\|\cdot\|^{2r}, X_k\right\rangle^2\right) \\ &\lesssim e^{-(k+2)\alpha}(1+\|x\|^{4r}) + e^{-2\alpha(k+1)}\mathbb{P}_{\delta_x}\left\langle 1+\|\cdot\|^{2r}, X_k\right\rangle^2.\end{aligned}$$

By(2.22) and (2.41), we get

$$\mathbb{P}_{\delta_x}\langle (1+\|\cdot\|^{2r}), X_k\rangle = e^{\alpha k}T_k(1+\|\cdot\|^{2r})(x) \lesssim e^{\alpha k}(1+\|x\|^{2r}),$$

$$\mathbb{V}ar_{\delta_x}\langle (1+\|\cdot\|^{2r}), X_k\rangle \lesssim e^{2\alpha k}(1+\|x\|^{4r}).$$

So we have

$$\mathbb{P}_{\delta_x}\langle (1+\|\cdot\|^{2r}), X_k\rangle^2 \lesssim e^{2\alpha k}(1+\|x\|^{4r}). \tag{3.56}$$

Thus $\sup_{k>0}\mathbb{P}_{\delta_x}|J_0(k)|^4 < \infty$ which implies $|J_0(k)|^2$ is uniformly integrable.

For $J_1(k)$, from the proof of (3.13), we see that (3.13) is also true when $\alpha = 2\gamma(f)b$. So we have $J_1(k) \overset{d}{\to} \sqrt{W_\infty}G$ where G is a Gaussian random variable. We also have $\mathbb{P}_{\delta_x}|J_1(k)|^2 \to \mathbb{P}_{\delta_x}W_\infty G^2$. Thus, $J_1(k)$ is uniformly integrable by [10, Theorem 5.5.2] and Skorokhod's representation theorem.

(ii) Recall that

$$A(m-k,m,s) \;=\; A_1(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2},\delta) \cup A_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2},\delta).$$

By Chebyshev's inequality

$$\mathbb{P}_{\delta_x}(A_1(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}),\delta) \;\le\; c^{-2}e^{-\alpha(m-k+1)}e^{-2\delta k}\mathbb{P}_{\delta_x}|S_k(T^\alpha_{m+s-k+1}f)|^2.$$

By (2.41) and (2.32), we have

$$\begin{aligned}&\mathbb{P}_{\delta_x}|S_k(T^\alpha_{m+s-k+1}f)|^2 = k^{-1}e^{-\alpha k}\mathbb{V}ar_{\delta_x}\langle T^\alpha_{m+s-k+1}f, X_k\rangle \\ =\;& Ak^{-1}e^{2\alpha(m+s-k+1)}\int_0^k e^{\alpha u}T_{k-u}[T_{u+m+s-k+1}f]^2(x)ds \lesssim e^{\alpha(m+s-k+1)}(1+\|x\|^{2r}),\end{aligned} \tag{3.57}$$

which implies

$$\sup_{k>m\epsilon}\sup_{0\le s<1}\mathbb{P}_{\delta_x}(A_1(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}),\delta) \lesssim \sup_{k>m\epsilon} e^{-2\delta k}(1+\|x\|^{2r}) \to 0, \quad m\to\infty. \tag{3.58}$$

It is easy to see that

$$A_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2},\delta) \subset \left\{|R(k, T^\alpha_{m+s-k}f)| > ce^{\alpha(m-k)/2}e^{\delta k}(e^\delta\sqrt{k+1}-\sqrt{k})\right\}. \tag{3.59}$$

Similarly, by Chebyshev's inequality, we have

$$\mathbb{P}_{\delta_x}(A_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}), \delta) \leq c^{-2}e^{-\alpha(m-k)}e^{-2\delta k}(e^\delta\sqrt{k+1}-\sqrt{k})^{-2}\mathbb{P}_{\delta_x}|R(k, T^\alpha_{m+s-k}f)|^2.$$

Using an argument similar to that leading to (3.50), we get

$$\mathbb{P}_{\delta_x}|R(k, T^\alpha_{m+s-k}f)|^2 = e^{-\alpha}T_k(\mathbb{V}ar_{\delta_\cdot}\langle T^\alpha_{m+s-k}f, X_1\rangle)(x),$$

so using an argument similar to that leading to (3.51), we obtain

$$\mathbb{V}ar_{\delta_x}\langle T^\alpha_{m+s-k}f, X_1\rangle \lesssim e^{\alpha(m-k)}(1+\|x\|^{2r}), \tag{3.60}$$

which implies $\mathbb{P}_{\delta_x}|R(k, T^\alpha_{m+s-k}f)|^2 \lesssim e^{\alpha(m-k)}(1+\|x\|^{2r})$. Thus

$$\sup_{k>t\epsilon}\sup_{0\leq s<1}\mathbb{P}_{\delta_x}(A_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2})) \lesssim \sup_{k>m\epsilon} e^{-2\delta k}(e^\delta\sqrt{k+1}-\sqrt{k})^{-2}(1+\|x\|^{2r}) \to 0, \tag{3.61}$$

as $m\to\infty$. Claim (ii) now follows easily from (3.58) and (3.61).

To finish the proof, we need to show that

$$\sup_{0\leq s<1} L_2(s,f,m) = \sup_{0\leq s<1}\frac{1}{m+1}\sum_{k=0}^{m} e^{-\alpha(m-k)}G_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}, \delta) \to 0, \quad m\to\infty. \tag{3.62}$$

By (3.59) and Chebyshev's inequality, we have

$$\begin{aligned}
& G_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}, \delta) \\
= \;& 2e^{-\alpha}k\mathbb{P}_{\delta_x}\Big(|S_k(T^\alpha_{m+s-k+1}f)|^2; A_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}, \delta)\Big) \\
\leq \;& 2e^{-\alpha}kce^{\alpha(m-k+1)/2}e^{\delta k}\mathbb{P}_{\delta_x}\Big(|S_k(T^\alpha_{m+s-k+1}f)|; A_2(k, T^\alpha_{m+s-k}f, ce^{\alpha(m-k)/2}, \delta)\Big) \\
\leq \;& 2kce^{\delta k+\alpha(m-k-1)/2}\mathbb{P}_{\delta_x}\Big(|S_k(T^\alpha_{m+s-k+1}f)|; |R(k, T^\alpha_{m+s-k}f)| > ce^{\delta k+\alpha/2(m-k)}(e^\delta\sqrt{k+1}-\sqrt{k})\Big) \\
\leq \;& 2c^{-1}e^{-\alpha(m-k+1)/2-\delta k}(e^\delta\sqrt{k+1}-\sqrt{k})^{-2}k\mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)||R(k, T^\alpha_{m+s-k}f)|^2\right) \\
\lesssim \;& e^{-\alpha(m-k)/2-\delta k}\mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)||R(k, T^\alpha_{m+s-k}f)|^2\right) \\
= \;& e^{-\alpha(m-k)/2-\delta k}e^{-\alpha(k+1)}\mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)|\langle \mathbb{V}ar_{\delta_\cdot}\langle T^\alpha_{m+s-k}f, X_1\rangle, X_k\rangle\right).
\end{aligned}$$

By (3.60), we get

$$\begin{aligned}
& \mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)|\langle \mathbb{V}ar_{\delta_\cdot}\langle T^\alpha_{m+s-k}f, X_1\rangle, X_k\rangle\right) \\
\lesssim \;& e^{\alpha(m-k)}\mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)|\langle(1+\|\cdot\|^{2r}), X_k\rangle\right) \\
\leq \;& e^{\alpha(m-k)}\sqrt{\mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)|^2\right)\mathbb{P}_{\delta_x}\langle(1+\|\cdot\|^{2r}), X_k\rangle^2}.
\end{aligned}$$

Thus by (3.57) and (3.56), we get

$$\mathbb{P}_{\delta_x}\left(|S_k(T^\alpha_{m+s-k+1}f)|\langle \mathbb{V}ar_{\delta_\cdot}\langle T^\alpha_{m+s-k}f, X_1\rangle, X_k\rangle\right) \lesssim e^{\alpha(m-k)}e^{\alpha(m+k)/2}(1+\|x\|^{3r}),$$

which implies

$$G_2(k, T^\alpha_{m+s-k} f, ce^{\alpha(m-k)/2}) \lesssim e^{\alpha(m-k)} e^{-\delta k}(1 + \|x\|^{3r}).$$

Therefore, we have

$$\sup_{0\le s<1} L_2(s, f, m) \lesssim \frac{1}{m+1} \sum_{k=0}^{m} e^{-\delta k}(1 + \|x\|^{3r}) \to 0, \quad m \to \infty. \tag{3.63}$$

Hence, $\lim_{t\to\infty} A_2(t) = 0$. □

In the following lemma we give a result similar to Lemma 3.2 for the process I.

Lemma 3.3 *Assume $f \in \mathcal{P}$ satisfies $\alpha = 2\gamma(f)b$. Define*

$$Y^*_t(f) := t^{-1/2} e^{-(\alpha/2)t} \left(\langle f, I_t\rangle - \mathbb{Q}_{\delta_x}\langle f, I_t\rangle\right).$$

Then for any $c > 0$ and $\delta > 0$, we have

$$\lim_{t\to\infty} \mathbb{Q}_{\delta_x} \left(|Y^*_t(f)|^2; |Y^*_t(f)| > ce^{\delta t} \right) = 0. \tag{3.64}$$

Proof: Recall the decomposition in (2.4). Define

$$S^*_t = t^{-1/2} e^{-(\alpha/2)t}(\langle f, \widetilde{X}_t\rangle - \mathbf{P}_{\delta_x}\langle f, \widetilde{X}_t\rangle),$$

$$S_t = t^{-1/2} e^{-(\alpha/2)t}(\langle f, \Lambda_t\rangle - \mathbf{P}_{\delta_x}\langle f, \Lambda_t\rangle),$$

and

$$\widetilde{Y}_t = t^{-1/2} e^{-(\alpha/2)t}(\langle f, I_t\rangle - \mathbf{P}_{\delta_x}\langle f, I_t\rangle).$$

Then we have $\widetilde{Y}_t = S_t - S^*_t$. Thus

$$\begin{aligned}
\mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t}) &\le \mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |S_t| > (c/2)e^{\delta t}) + \mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |S^*_t| > (c/2)e^{\delta t}) \\
&\le 2\mathbf{P}_{\delta_x}(|S_t|^2; |S_t| > (c/2)e^{\delta t}) + 2\mathbf{P}_{\delta_x}(|S^*_t|^2) + \mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |S^*_t| > (c/2)e^{\delta t}) \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

By Lemma 3.2, we have $\lim_{t\to\infty} I_1(t) = 0$. By (2.59), we have

$$I_2(t) = 2t^{-1} e^{-\alpha t} \mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_t\rangle \to 0, \quad t \to \infty.$$

Since I_t and $\widetilde{X}$ are independent, we have

$$I_3(t) = \mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2)\ \mathbf{P}_{\delta_x}(|S^*_t| > (c/2)e^{\delta t}).$$

Since $S_t = S^*_t + \widetilde{Y}_t$, and S^*_t and $\widetilde{Y}_t$ are independent, by (2.38), we get

$$\mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2) = \mathbf{P}_{\delta_x}(|S_t|^2) - \mathbf{P}_{\delta_x}(S^*_t|^2) \to \rho^2_f, \quad t \to \infty.$$

By Chebyshev's inequality, we have

$$\mathbf{P}_{\delta_x}(|S_t^*| > (c/2)e^{\delta t}) \le (c/2)^{-2}e^{-2\delta t})\mathbf{P}_{\delta_x}(|S_t^*|^2) \to 0, \quad t \to \infty.$$

Hence $\lim_{t\to\infty} I_3(t) = 0$. Thus

$$\mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t}) \to 0. \tag{3.65}$$

Recall that under $\mathbf{P}_{\delta_x}$, $I_t = \sum_{j=1}^N I_t^j$, where $I^j, j = 1, ...$ are independent copies of I under $\mathbb{Q}_{\delta_x}$, and are independent of N. Thus,

$$\mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t}) \ge \mathbf{P}_{\delta_x}(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t}, N = 1) = \mathbf{P}_{\delta_x}(N = 1)\mathbb{Q}_{\delta_x}\left(|Y_t^*(f)|^2; |Y_t^*(f)| > ce^{\delta t}\right).$$

Since $\mathbf{P}_{\delta_x}(N = 1) > 0$, (3.64) follows easily from (3.65). $\square$

Now, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5: The proof is similar to that of Theorem 1.3. We define an $\mathbb{R}^2$-valued random variable by

$$U_1(t) := (e^{-\alpha(t)} \|\Lambda_t\|, t^{-1/2}e^{-(\alpha/2)(t)}\langle f, \Lambda_t\rangle).$$

We need to show that as $t \to \infty$,

$$U_1(t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty}G_2(f)), \tag{3.66}$$

where $G_2(f) \sim \mathcal{N}(0, \rho_f^2)$. Let $n > 0$ and write

$$U_1(nt) = (e^{-\alpha(nt)} \|\Lambda_{nt}\|, (nt)^{-1/2}e^{-(\alpha/2)(nt)}\langle f, \Lambda_{nt}\rangle).$$

Recall the representation (3.1). Define

$$Y_t^{u,n} := ((n-1)t)^{-1/2}e^{-\alpha(n-1)t/2}\langle f, I_{(n-1)t}^{u,t}\rangle \quad \text{and} \quad y_t^{u,n} := \mathbf{P}_\mu(Y_t^{u,n}|\mathcal{G}_t).$$

$Y_t^{u,n}$ has the same distribution as $Y_t^n := ((n-1)t)^{-1/2}e^{-\alpha(n-1)t/2}\langle f, I_{(n-1)t}\rangle$ under $\mathbb{Q}_{\delta_{Z_u(t)}}$. Thus

$$\begin{aligned}
& (nt)^{-1/2}e^{-(\alpha/2)nt}\langle f, \Lambda_{nt}\rangle \\
= \quad & (nt)^{-1/2}e^{-(\alpha/2)nt}\langle f, \widetilde{X}_{(n-1)t}^t\rangle + \sqrt{\frac{n-1}{n}}e^{-(\alpha/2)t}\sum_{u\in\mathcal{L}_t} Y_t^{u,n} \\
= \quad & (nt)^{-1/2}e^{-(\alpha/2)nt}(\langle f, \widetilde{X}_{(n-1)t}^t\rangle - \mathbf{P}_\mu(\langle f, \widetilde{X}_{(n-1)t}^t\rangle|\mathcal{G}_t) + \sqrt{\frac{n-1}{n}}e^{-(\alpha/2)t}\sum_{u\in\mathcal{L}_t}(Y_t^{u,n} - y_t^{u,n}) \\
& +(nt)^{-1/2}e^{-(\alpha/2)nt}\mathbf{P}_\mu(\langle f, \Lambda_{nt}\rangle|\mathcal{G}_t) \\
=: \quad & J_0^n(t) + J_1^n(t) + J_2^n(t).
\end{aligned} \tag{3.67}$$

Put $\widetilde{V}_s(x) := \mathbf{V}ar_{\delta_x}\langle f, \widetilde{X}_s\rangle$. Then by (2.59), there exists $r \in \mathbb{N}$ such that $\widetilde{V}_s(x) \lesssim e^{-\alpha^* s}(1 + \|x\|^{2r})$. From the definition of $\widetilde{X}_s^t$, we have

$$\mathbf{P}_\mu J_0^n(t)^2 \quad = \quad (nt)^{-1}e^{-\alpha(nt)}\mathbf{P}_\mu(\langle\widetilde{V}_{(n-1)t}, \Lambda_t\rangle) = (nt)^{-1}e^{-\alpha(n-1)t}\langle T_t(\widetilde{V}_{(n-1)t}, \mu\rangle$$

$$\lesssim \quad (nt)^{-1}e^{-\alpha(n-1)t}e^{-\alpha^*(n-1)t} \to 0, \quad \text{as } t\to\infty. \tag{3.68}$$

Using an argument similar to that in the proof of Theorem 1.3, we can get

$$\begin{aligned}\mathbf{P}_\mu J_2^n(t)^2 &= A(nt)^{-1}e^{\alpha(n-1)t}\int_{\mathbb{R}^d}\int_0^t e^{\alpha u}T_{t-u}[T_{(n-1)t+u}f]^2(x)\,ds\,\mu(dx)\\ &\lesssim n^{-1}.\end{aligned} \tag{3.69}$$

Combining (3.68) and (3.69), there exists $c>0$ such that

$$\limsup_{t\to\infty}\mathbf{P}_\mu(J_0^n(t)+J_2^n(t))^2 \le c/n. \tag{3.70}$$

Now we consider $J_1^n(t)$. We define an $\mathbb{R}^2$-valued random variable $U_2(n,t)$ by

$$U_2(n,t) := \left(e^{-\alpha t}\|\Lambda_t\|, e^{-(\alpha/2)t}\sum_{u\in\mathcal{L}_t}(Y_t^{u,n}-y_t^{u,n})\right).$$

We claim that

$$U_2(n,t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty}G_2(f)), \qquad \text{as } t\to\infty. \tag{3.71}$$

Denote the characteristic function of $U_2(n,t)$ under $\mathbf{P}_\mu$ by $\kappa_2(\theta_1,\theta_2,n,t)$. Using an argument similar to that leading to (3.14), we get

$$\kappa_2(\theta_1,\theta_2,n,t) = \mathbf{P}_\mu\left(\exp\{i\theta_1 e^{-\alpha t}\|\Lambda_t\|\}\exp\left\{\lambda^*\langle h_t^n(\cdot,\ e^{-(\alpha/2)t}\theta_2)-1,\Lambda_t\rangle\right\}\right),$$

where $h_t^n(x,\theta)=\mathbb{Q}_{\delta_x}e^{i\theta(Y_t^n-\mathbb{Q}_{\delta_x}Y_t^n)}$. Define

$$e_t^n(x,\theta) := h_t^n(x,\theta)-1+\frac{1}{2}\theta^2\mathbb{V}_{\delta_x}Y_t^n$$

and $V_t^n(x) := \mathbb{V}_{\delta_x}Y_t^n$. Then

$$\begin{aligned}&\exp\left\{\lambda^*\langle h_t^n(\cdot,\ e^{-(\alpha/2)t}\theta_2)-1,\Lambda_t\rangle\right\}\\ =\ &\exp\left\{-\frac{1}{2}\lambda^*\theta_2^2e^{-\alpha t}\langle V_t^n,\Lambda_t\rangle\right\}\exp\left\{\lambda^*\langle e_t^n(\cdot,e^{-(\alpha/2)t}\theta_2),\Lambda_t\rangle\right\}\\ =:\ &J_{1,1}(n,t)J_{1,2}(n,t).\end{aligned}$$

We first consider $J_{1,1}(n,t)$. By (2.37), we have that as $t\to\infty$,

$$e^{-\alpha t}\langle|\lambda^*V_t^n-\rho_f^2|,\Lambda_t\rangle \lesssim t^{-1}e^{-\alpha t}\langle(1+\|x\|^r),\Lambda_t\rangle \to 0 \quad \text{in probability}.$$

It follows that

$$\lim_{t\to\infty}e^{-\alpha t}\langle\lambda^*V_t^n,\Lambda_t\rangle = \lim_{t\to\infty}e^{-\alpha t}\langle\rho_f^2,\Lambda_t\rangle = \rho_f^2W_\infty \quad \text{in probability}, \tag{3.72}$$

which implies that $\lim_{t\to\infty} J_{1,1}(n,t) = \exp\left\{-\frac{1}{2}\theta_2^2\rho_f^2 W_\infty\right\}$.

For $J_{1,2}(n,t)$, by (3.5), we have, for any $\epsilon > 0$,

$$\begin{aligned}
|e_t^n(x, e^{-(\alpha/2)t}\theta_2)| &\leq \frac{1}{6}|\theta_2|^3 e^{-\frac{3}{2}\alpha t}\mathbb{Q}_{\delta_x}\left(|Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n|^3; |Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n| < \epsilon e^{\alpha t/2}\right) \\
&\quad + \theta_2^2 e^{-\alpha t}\mathbb{Q}_{\delta_x}\left(|Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n|^2; |Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n| \geq \epsilon e^{\alpha t/2}\right) \\
&\leq \frac{\epsilon}{6}|\theta|_2^3 e^{-\alpha t}\mathbb{Q}_{\delta_x}\left(|Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n|^2\right) \\
&\quad + \theta_2^2 e^{-\alpha t}\mathbb{Q}_{\delta_x}\left(|Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n|^2; |Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n| \geq \epsilon e^{\alpha t/2}\right) \\
&= \frac{\epsilon}{6}|\theta|_2^3 e^{-\alpha t}V_t^n(x) + \theta_2^2 e^{-\alpha t}F_t^n(x),
\end{aligned}$$

where $F_t^n(x) = \mathbb{Q}_{\delta_x}\left(|Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n|^2; |Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n| \geq \epsilon e^{\alpha t/2}\right)$. It follows from Lemma 3.3 that $\lim_{t\to\infty} F_t^n(x) = 0$. By (2.53), we also have

$$F_t^n(x) \leq \mathbb{Q}_{\delta_x}\left(|Y_t^n - \mathbb{Q}_{\delta_x}Y_t^n|^2\right) \lesssim 1 + \|x\|^{2r}.$$

Note that

$$e^{-\alpha t}\mathbf{P}_\mu\langle F_t^n(x), \Lambda_t\rangle = \langle T_t(F_t^n), \mu\rangle.$$

Thus by the dominated convergence theorem, we get $\lim_{t\to\infty} e^{-\alpha t}\mathbf{P}_\mu\langle F_t^n(x), \Lambda_t\rangle = 0$. It follows that $e^{-\alpha t}\langle F_t^n(x), \Lambda_t\rangle \to 0$ in probability. Furthermore from (3.72), we obtain that as $t \to \infty$,

$$\frac{\epsilon}{6}\theta_2^3 e^{-\alpha t}\langle V_t^n, \Lambda_t\rangle \to \frac{\epsilon}{6\lambda^*}\theta_2^3\rho_f^2 W_\infty \quad \text{in probability.}$$

Thus, letting $\epsilon \to 0$, we get that as $t \to \infty$,

$$\langle |e_t^n(x, e^{-(\alpha/2)t}\theta_2)|, \Lambda_t\rangle \to 0 \quad \text{in probability,} \tag{3.73}$$

which implies $J_{1,2}(n,t) \to 1$ in probability, as $t \to \infty$.

Thus, when $t \to \infty$,

$$\exp\left\{\lambda^*\langle h_t^n(\cdot,\ e^{-(\alpha/2)t}\theta_2) - 1, \Lambda_t\rangle\right\} \to \exp\left\{-\frac{1}{2}\theta_2^2\rho_f^2 W_\infty\right\} \tag{3.74}$$

in probability. Since $h_t^n(x,\theta)$ is a characteristic function, its real part is less than 1, which implies

$$|\exp\left\{\lambda^*\langle h_t^n(\cdot,\ e^{-(\alpha/2)t}\theta_2) - 1, \Lambda_t\rangle\right\}| \leq 1.$$

So by the dominated convergence theorem, we get that

$$\lim_{t\to\infty} \kappa_2(\theta_1, \theta_2, n, t) = \mathbf{P}_\mu \exp\{i\theta_1 W_\infty\}\exp\left\{-\frac{1}{2}\theta_2^2\rho_f^2 W_\infty\right\}, \tag{3.75}$$

which implies our claim (3.71). By (3.71), we easily get that as $t \to \infty$,

$$U_3(n,t) \quad := \quad \left(e^{-\alpha(nt)}\|\Lambda_{nt}\|, J_1^n(t)\right) \xrightarrow{d} (W_\infty, \sqrt{\frac{n-1}{n}}\sqrt{W_\infty}G_2(f)).$$

Let $\mathcal{L}(nt)$ and $\widetilde{\mathcal{L}}^n(t)$ be the distributions of $U_1(nt)$ and $U_3(n,t)$ respectively, and let $\mathcal{L}^n$ and $\mathcal{L}$ be the distributions of $(W_\infty, \sqrt{\frac{n-1}{n}}\sqrt{W_\infty}G_2(f))$ and $(W_\infty, \sqrt{W_\infty}G_2(f))$ respectively. Then, using (3.4), we have

$$\begin{aligned}\limsup_{t\to\infty} \beta(\mathcal{L}(nt), \mathcal{L}) \quad &\le \quad \limsup_{t\to\infty}[\beta(\mathcal{L}(nt), \widetilde{\mathcal{L}}^n(t)) + \beta(\widetilde{\mathcal{L}}^n(t), \mathcal{L}^n) + \beta(\mathcal{L}^n, \mathcal{L})] \\ &\le \quad \limsup_{t\to\infty}(\mathbf{P}_\mu(J_0^n(t) + J_2^n(t))^2)^{1/2} + 0 + \beta(\mathcal{L}^n, \mathcal{L}). \end{aligned} \tag{3.76}$$

Using this and the definition of $\limsup_{t\to\infty}$, we easily get that

$$\limsup_{t\to\infty} \beta(\mathcal{L}(t), \mathcal{L}) = \limsup_{t\to\infty} \beta(\mathcal{L}(nt), \mathcal{L}) \le \sqrt{c/n} + \beta(\mathcal{L}^n, \mathcal{L}).$$

Letting $n \to \infty$, we get $\limsup_{t\to\infty} \beta(\mathcal{L}(t), \mathcal{L}) = 0$. The proof is now complete. $\square$

Proof of Theorem 1.12: First note that

$$\begin{aligned} & t^{-1/2}\|X_t\|^{-1/2}\left(\langle f, X_t\rangle - \sum_{\gamma(f)\le m<\alpha/2b} e^{(\alpha-mb)t}\sum_{|p|=m} a_p H_\infty^p\right) \\ = \quad & t^{-1/2}\|X_t\|^{-1/2}\langle f_{(cl)}, X_t\rangle + t^{-1/2}\|X_t\|^{-1/2}\left(\langle f_{(s)}, X_t\rangle - \sum_{n=1}^{k} e^{(\alpha-mb)t}\sum_{|p|=m} a_p H_\infty^p\right) \\ =: \quad & J_1(t) + J_2(t), \end{aligned}$$

where $f_{(cl)} = f_{(l)} + f_{(c)}$. By the definition of $f_{(s)}$, we have $(f_{(s)})_{(c)} = 0$. Then using Theorem 1.9 for $f_{(s)}$, we have

$$\|X_t\|^{-1/2}\left(\langle f_{(s)}, X_t\rangle - \sum_{n=1}^{k} e^{(\alpha-mb)t}\sum_{|p|=m} a_p H_\infty^p\right) \xrightarrow{d} G_1(f_{(s)}). \tag{3.77}$$

Thus

$$J_2(t) \xrightarrow{d} 0, \quad t \to \infty. \tag{3.78}$$

Since $\alpha = 2\gamma(f_{(cl)})b$, so using Theorem 1.5 for $f_{(cl)}$, we have

$$(e^{-\alpha t}\|X_t\|, J_1(t))) \xrightarrow{d} (W^*, G_2(f_{(cl)})), \tag{3.79}$$

where $G_2(f_{(cl)}) \sim \mathcal{N}(0, \rho^2_{f_{(cl)}})$. By (1.17), we have $\rho^2_{f_{(cl)}} = A\sum_{|p|=\alpha/2b}(a_p)^2$. Combing (3.78) and (3.79), we arrive at the conclusion of Theorem 1.12. $\square$

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